ANALYTICAL SOLUTION OF CASSON FLUID BOUNDARY LAYER FLOW AND HEAT TRANSFER DUE TO STRETCHING SHEET

MAHANTESH M. NANDEPPANAVAR*
Department of Mathematics,
Government College (Autonomous), Kalaburgi-585105, Karnataka, INDIA.

ELISHA DEVDAS
Department of Mathematics,
Government College, Mudgal, Karnataka, INDIA.

(Received On: 27-05-19; Revised & Accepted On: 24-06-19)

ABSTRACT

In this chapter we have studied flow and heat transfer characteristics of Non-Newtonian casson fluid over an inclined stretching sheet. The governing equations which are derived from the Navier-Stokes equations of motion and Prandtl boundary layer concept respectively. The governing equations of motion and heat transfer are basically Non-linear partial differential equations which are difficult to solve, hence we have converted these partial differential equations into Non-linear ordinary differential equations by using suitable Non-dimensionalization with respective boundary conditions. These ordinary differential equations are solved numerically using Ranga-Kutta fourth order method with an efficient shooting technique. Using the drawn graph, we have discussed the results of the present study. Numerical values of skin friction and Nusselt numbers are calculated and tabulated. Our results are in good agreement with the earlier published data for some limiting conditions.

Key Words: Flow, Heat Transfer, Prandtl number, Runge-kutta Method, Nusselt number.

1. INTRODUCTION

The flow of viscous incompressible fluid over stretching sheet is one of the most applicable flows in manufacturing industries and technological process. Such as glass fiber production, wire drawing, paper production, plastic sheets, glass blowing, artificial fibers, hot rolling, metal and polymer processing industries and others. Hayat et al. [1] have studied the mixed convection stagnation point flow of an incompressible fluid flow over stretching sheet under convective boundary conditions. Pramanik [2] has investigated the boundary layer flow of Non-Newtonian fluid and heat transfer due to an exponentially stretching surface in the presence of suction or blowing. Swati Mukhopadlyaya et al. [3] have examined the boundary layer flow of Non-Newtonian fluid of an exponential stretching surface in the presence of an applied magnetic field. Mahdy and Chamkaha [4] have studied the unsteady two dimensional flow of Non-Newtonian nano fluid over a contracting cylinder using Buongiorno’s model and casson fluid model. Sarojimma, and Vasundhra [5] have investigated the flow, heat and mass transfer characteristics of magneto hydrodynamic casson fluid in a parallel plate channel with stretching walls due to a uniform transverse magnetic field. Mahdy [6] has investigated the effect of slip at the boundary of unsteady two dimensional magneto hydro dynamic flow of Non-Newtonian fluid over stretching surface in the presence of suction or blowing. Emmanuel Maurice et al. [7] have examined the flow of casson fluid over a vertical porous surface with chemical reaction in the presence of magnetic field. Mustafa and Junaid [8] have presented the magneto hydrodynamic flow and heat transfer of casson nano fluid over a non-linearly stretching sheet. Hayat et al. [9] have reported the effect of inclined magnetic field and heat transfer in the flow of a third grade fluid of an exponentially stretching surface. Sathies and Gangadhar [10] have studied two dimensional magneto hydro dynamic stagnation point flow of electrically conducting Non-Newtonian casson fluid and heat transfer due to a stretching sheet in the presence of mass transfer and chemical reaction. Kalaivanan et al. [11] have reported the effect of inclined magnetic field on slip flow of casson fluid over stretching sheet. Sarojamma and Vendabai [12] have examined the effect of magnetic field and heat resource on the steady boundary layer flow and heat transfer of casson nano-fluid on a vertical cylinder stretching exponentially. Sugunamma et al. [13] have studied an unsteady free convection flow of casson fluid bounded by a moving vertical flat plate in a rotating system with convective boundary conditions. Akbar et al. [14] have examined magneto hydrodynamic flow and heat transfer of
electrically conducting viscoplastic fluids through a channel. Saidulu and Ventaka [15] have presented the boundary layer flow of a Non-Newtonian casson fluid accompanied by heat and mass transfer due to a porous exponentially stretching sheet. Bala [16] has investigated the steady two dimensional magneto hydrodynamic boundary layer flow of casson fluid over an exponentially stretching surface in the presence of thermal radiation and chemical reaction. Ajayi et al. [17] have studied the effects of viscous dissipation on the motion of casson fluid over an upper horizontal thermal stratified melting surface of paraboloid revolution. Pushpalatha et al. [18] have reported the effects of unsteady magneto hydrodynamic casson fluid flow towards a stretching surface with cross diffusion. Vajravelu et al. [19] have investigated a coupled Non-linear boundary value problem obtained from a mixed convective flow of a Non-Newtonian fluid at a vertical stretching sheet with variable thermal conductivity. Adamu et al. [20] have studied the effect of mass transfer on the magneto hydrodynamic flow of the casson fluid through a porous media of a stretching surface with chemical reaction and suction. Sharada and Shankar [21] have investigated the effects of different parameters of the mixed convection partial slip flow of a casson fluid over a vertically stretching sheet with convective boundary conditions. Eswara and Sreenadh [22] have studied the steady two dimensional magneto hydrodynamic convective boundary layer flow of a casson fluid over a permeable stretching surface in the presence of thermal radiation and chemical reaction. Gireesha et al. [23] have investigated the effect of Non-uniform heat generation/absorption in heat and mass transfer for magneto hydrodynamic casson fluid boundary layer flow over a permeable stretching sheet through a porous medium.

In light of the above literatures, it is evident that numerous investigations have been conducted on the flow and heat transfer of casson fluid due to exponentially stretching sheet. However to the best of the author’s knowledge the problem of flow and heat transfer of casson fluid due to an exponential inclined stretching sheet, has not yet addressed. In the present paper, we therefore considered flow and heat transfer of casson fluid due to exponential inclined stretching sheet. The sheets are stretched along the x-axis by the action of two equal and opposite forces as shown in the figure below with velocity \( u(x) = U_o e^{\frac{x}{L}} \). The Sheet is considered to vary as an exponential function of the distance from the origin where the slit is situated. Here L is the reference length and \( U_o \) is the reference velocity.

Governing equations of Non-Newtonian liquid over an inclined stretching sheet are given as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (1)
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu (1 + \frac{1}{\beta}) \frac{\partial^2 u}{\partial y^2} + g\beta' (T - T_\infty) \sin \alpha \quad (2)
\]

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \quad (3)
\]

Boundary conditions used are:

\[
\begin{align*}
    &u = U_w(x) = U_o e^{\frac{x}{L}}, \quad v = 0, \quad T = T_w = T_\infty + Ae^{\frac{2x}{L}} \text{ at } y = 0 \\
    &u \rightarrow 0, \quad T \rightarrow \infty \text{ as } y \rightarrow \infty
\end{align*}
\]

(4)

The dimensionless variables used are:

\[
(X, Y) = \left( \frac{x, y \sqrt{Re}}{L} \right), \quad (U, V) = \left( \frac{u, v \sqrt{Re}}{U_o} \right), \quad \bar{T} = \frac{T - T_\infty}{T_w - T_\infty}
\]

(5)

\[
\text{Re} = \frac{U_o L}{D}, \quad \text{Reynolds number}
\]
\[ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \]  \hspace{1cm} (6)

\[ U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 U}{\partial Y^2} + e^{2x} Gr \sin \alpha \bar{T} = 0 \]  \hspace{1cm} (7)

\[ U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial Y^2} - 2U \bar{T} \]  \hspace{1cm} (8)

The transformed boundary conditions are:

\[ U = e^x, \ V = 0, \ \bar{T} = 1 \hspace{1cm} \text{at} \ Y = 0 \]
\[ U \rightarrow 0, \ \bar{T} \rightarrow 0 \hspace{1cm} \text{as} \ Y \rightarrow \infty \]  \hspace{1cm} (9)

Introducing Stream function \( \psi(X, Y) \) as:

\[ U = \frac{\partial \psi}{\partial Y}, \ V = -\frac{\partial \psi}{\partial X} \]  \hspace{1cm} (10)

The converted equations in terms of \( \psi \) are:

\[ T \left( 1 + \frac{1}{\beta} \right) \frac{\partial^3 \psi}{\partial Y^3} + \frac{\partial}{\partial (X,Y)} \left( \psi, \frac{\partial \psi}{\partial Y} \right) + e^{2x} Gr \sin \alpha \bar{T} = 0 \]  \hspace{1cm} (11)

\[ \frac{\partial^2 \bar{T}}{\partial Y^2} + Pr \frac{\partial (Y, \bar{T})}{\partial (X,Y)} - 2Pr \frac{\partial \bar{T}}{\partial Y} \bar{T} = 0 \]  \hspace{1cm} (12)

The boundary conditions in terms of \( \psi \) are:

\[ \frac{\partial \psi}{\partial Y} = e^x, \ \frac{\partial \psi}{\partial X} = 0, \ \bar{T} = 1 \hspace{1cm} \text{at} \ Y = 0 \]
\[ \frac{\partial \psi}{\partial Y} \rightarrow 0, \ \bar{T} \rightarrow 0 \hspace{1cm} \text{as} \ Y \rightarrow \infty \]  \hspace{1cm} (13)

The partial differential equations in terms of \( \psi \) are converted into ordinary differential equations using the following similarity transformations:

\[ \psi(X, Y) = \sqrt{2e^x} f(\eta), \ \bar{T}(X, Y) = \theta(\eta) \]

Where \( \eta = \frac{Y}{\sqrt{2} e^x} \)  \hspace{1cm} (14)

The converted Ordinary differential equations with respective boundary conditions are:

\[ \left( 1 + \frac{1}{\beta} \right) f'''' + ff'' - 2f'^2 + 2Gr \sin \alpha \theta = 0 \]

\[ \theta'' + Pr (f \theta' - 4 f' \theta) = 0 \]  \hspace{1cm} (15)

\[ f(0) = 0, \ \ f'(0) = 1, \ \ \theta(0) = 1 \]
\[ f'(\infty) \rightarrow 0, \ \ \theta(\infty) \rightarrow 0 \]  \hspace{1cm} (16)
Skin friction:

\[ C_f = \frac{\tau_w}{\rho u_w^2} \]

It is given by Where \( \tau_w = -\mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = -\frac{\mu U_e \sqrt{Re} \ e^2}{L \sqrt{2}} f''(0) \)

\[ C_f = \frac{-\sqrt{2}}{\sqrt{Re}_x} f''(0) \] \hspace{1cm} (17)

Nusselt number:

The local heat transfer rate from surface of stretching sheet into cooling liquid is given by:

\[ q_w = -k \left[ \frac{\partial T}{\partial y} \right]_{y=0} = -k \frac{(T_w - T_\infty)}{L} \sqrt{Re}_x \ e^2 \theta'(0) \]

Local Nusselt number is given by:

\[ Nu_x = \frac{x q_w}{k(T_w - T_\infty)} = -1 \left( \frac{x}{L} \right) \sqrt{Re}_x \theta'(0) \]

\[ Re = \frac{u_w L}{\nu} \] is the local Reynolds number.

3. Method of Solution

The governing ordinary differential equation (15) to (16) are converted into an equivalent initial value problem and guessing the missing conditions

\[ f''(0) = \beta_1^* \text{ and } \theta'(0) = \beta_2^* \]

We represent the new series solution of this problem in the form of series which is in terms of \( 1 - e^{-\eta} \).

Let us choose

\[ Z = 1 - e^{-\eta} \] \hspace{1cm} (19)

Substituting (19) in equation (15) and (16) we get the ordinary differential equations with the boundary conditions in terms of \( Z \) as:

\[ \begin{cases} 1 + \frac{1}{\beta} (1-z)^3 \frac{d^3 f}{dz^3} + (1-z)^2 (3-f) \frac{d^2 f}{dz^2} - (1-z)(f-1) = 0 \\
\frac{df}{dz} - 2(1-z)^2 \left( \frac{df}{dz} \right)^2 + Gr \sin \alpha \theta = 0 \\
(1-z)^2 \frac{d^2 \theta}{dz^2} + (1-z)(1-Pr_f) \frac{d\theta}{dz} - 4 Pr(1-z) \frac{df}{dz} \theta = 0 \end{cases} \] \hspace{1cm} (20)

The initial conditions are:

\[ f(z) = 0, \quad \frac{df}{dz}(z) = 1, \quad \theta(z) = 1, \text{ at } z = 0 \] \hspace{1cm} (21)
The initial conditions are also converted as:
\[
\frac{d^2 f}{dz^2} = \beta_1, \quad \frac{d\theta}{dz} = \beta_2, \quad z = 0
\]
(22)

\(\beta_1\) and \(\beta_2\) are obtained using Newton-Raphson method.

The new series solutions of equation (20) are assumed as:
\[
f(z) = \sum_{k=0}^{\infty} A_k z^k \\
\theta(z) = \sum_{k=0}^{\infty} B_k z^k
\]
(23)

\(A_k\)'s and \(B_k\)'s are to be determined using initial conditions (21) and (22).

It is possible to have series solution in \(z\) due to the fact that 
\[|z| = |1 - e^{-\eta}| < 1.\]

Substituting (23) in (20) and equating the coefficients of various powers of \(z\) to zero we get a recursive relation for \(A_k\)'s and \(B_k\)'s and using the initial conditions we get the first few coefficients as:
\[A_0 = 1, \quad A_1 = \frac{\beta}{2!}, \quad B_0 = 1, \quad B_1 = \beta_2\]
(24)

The remaining coefficients are obtained using (24) and recursive relation.

Now reverting back to \(\eta\) as:
\[
f(\eta) = \sum_{k=0}^{\infty} A_k (1 - e^{-\eta})^k \\
\theta(\eta) = \sum_{k=0}^{\infty} B_k (1 - e^{-\eta})^k
\]
(25)

This is new series solution of considered problem.

4. RESULTS AND DISCUSSIONS

Here we have obtained NSS (new series solution) for the inclined stretching sheet /shrinking sheet problem in the powers of \(1 - e^{-\eta}\). This solution has good convergence because of the fact that \[|1 - e^{-\eta}| < 1.\]

This new series solution is also compared with numerical Runge-Kutta fourth order method with shooting technique, which found a very good agreement.

We now discuss the results pertaining to the inclined stretching sheet problem from the fluid dynamics point of view. Figure 2 and 3 show the effect of inclination angle on flow and heat transfer. Plot 2 reveals that as the angle of inclination increases from 0 to \[\frac{\pi}{2}\] the momentum boundary layer thickness increases, i.e when the inclination is increased the gravity effect becomes prominent which assist the liquid to flow freely. The inclination reduces the thermal boundary layer thickness as shown in Fig. 3. The results clearly reveal that inclination of the trenching can be effectively used a desired temperature between that seen in the horizontal \(\gamma = 0\) and vertical \(\gamma = \frac{\pi}{2}\) stretching sheet problems.

The impact of convection over the flow is depicted in Figs. 4 and 5 through the variation of the Grashof number \(Gr\). The effect of the Grashof number is identical to that of the inclination angle on the flow and heat transform. The graph 4 indicates that the momentum boundary layer thickness increases with increasing value of \(Gr\) enabling more flow. The buoyancy force that evolved as a consequence of cooling of the inclined stretching sheet acts like a favorable pressure gradient accelerating the liquid in the boundary layer region. It is evident from the plot 5 that increasing value of \(Gr\) in thinning of the thermal boundary layer associated with an increase in the wall temperature gradient and hence produces an increase in the heat transfer.
The effect of Prandtl number Pr on flow and heat transfer is projected in Figs. 6 and 7. Fig.6 clearly illustrates that increasing values of Pr reduces the horizontal velocity and Fig.7 indicates that large values of Prandtl number results in thinning of the thermal boundary layer.

The inclination angle is to reduce both skin friction at the wall and the Nusselt number. The effect of Prandtl number on these parameters is the opposite. In the numerical solution of the BVP there is a distinct problem that arises regarding the choice of ‘∞’. The chosen infinity and the actual infinity the numerical computation cannot generally coincide. It is the common experience of all numerical investigators that the results between the chosen infinity and the actual infinity are not exactly zero through negligibly small. This is typical of the stretching sheet problem as well. DTM with pade’ approximation reflects the correct results beyond infinity. In the absence of pade’ approximation any numerical method which investigators use for solving the BVP yields not so accurate results beyond the actual infinity. It is on this reason that we use the DTM with pade’ approximation. It is to be noted that the series solution obtained for the GFS equation can be put, on expanding the powers of \( e^{-\eta} \) and rearranging in the following form

\[
f(\eta) = \sum_{k=0}^{\infty} A_k e^{-k\eta}, \quad \theta(\eta) = \sum_{k=0}^{\infty} B_k e^{-k\eta}
\]

5. REFFERENCE

1. This new series solution is very simple way to find solution for the inclined stretching sheet problem.
2. The inclination of \( \alpha \) can be efficiently used to have a desired temperature in the system.
3. The effect of parameters \( \alpha, Gr, \) and \( Pr \) on the dynamics of cooling liquid are qualitatively similar but except \( Pr \), which has opposite effect

Figure-1: Schematic of the inclined stretching sheet problem.
Figure-2: variation of $f'(\eta)$ with inclination $\gamma$.

Figure-3: variation of $\theta(\eta)$ with the inclination $\gamma$.

Figure-4: variation of $f'(\eta)$ with Grashof number $Gr$. 

© 2019, IJMA. All Rights Reserved
Figure-5: variation of $\theta(\eta)$ with Gr

Figure-6: variation of $f'(\eta)$ with P.

Figure-7: variation of $\theta(\eta)$ with Prandtl number Pr.
Figure-8: Variation of $f' (\eta)$ with $\beta$.

Figure-9: Variation of $\theta (\eta)$ with $\beta$.

Source of support: Nil, Conflict of interest: None Declared.

[Copyright © 2019. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]