ZERO-FREE REGIONS FOR POLYNOMIALS WITH SPECIAL COMPLEX COEFFICIENTS

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(Received On: 10-03-20; Revised & Accepted On: 11-04-20)

ABSTRACT

In this paper we can extend the well-known result Eneström-Kakeya theorem by relaxing the hypothesis in several ways and obtain zero-free regions for polynomials with special complex coefficients and thereby present some interesting generalizations and extensions of the Eneström-Kakeya Theorem.

Mathematics Subject Classification: 30C10, 30C15.

Keywords: Zeros of polynomial, Polar Derivatives, Eneström-Kakeya theorem.

1. INTRODUCTION

The well-known Results Eneström-Kakeya theorem [2, 4] in theory of the distribution of zeros of polynomials is the following.

Theorem 1.1: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \) then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \).

Applying the above result to the polynomial \( z^n P\left(\frac{1}{z}\right) \) we get the following result:

Theorem 1.2: If \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_n \leq a_{n-1} \leq a_{n-2} \leq \ldots \leq a_0 \) then \( P(z) \) does not vanish in \( |z| < 1 \).

In the literature [1, 3, 5-9], there exist several extensions and generalizations of the Eneström-Kakeya Theorem. Recently B. A. Zargar [11] proved the following results:

Theorem 1.3: If \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that for some \( k \geq 1 \), \( 0 < a_n \leq a_{n-1} \leq a_{n-2} \leq \ldots \leq a_0 \) then \( P(z) \) does not vanish in the disk \( |z| < \frac{1}{2^{k-1}} \).

Theorem 1.4: If \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that for some real number \( \rho \geq 0 \)
\[ 0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n + \rho, \]
then \( P(z) \) does not vanish in the disk \( |z| < \frac{1}{2(ka_n + \rho) + a_n} \).

The following results due to P. Ramulu [10].

Theorem 1.5: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with real coefficients such that for some \( k \geq 1 \)
\[ \rho \geq 0, \quad a_m \neq 0, \quad a_n - \rho \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq \ldots \geq a_0 \geq a_0 \]
then all the zeros of \( P(z) \) does not vanish in the disk \( |z| < \frac{1}{2(ka_n + a_m)} \).

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Theorem 1.6: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with real coefficients such that for some \( 0 < r \leq 1, \rho \geq 0, a_m \neq 0, a_i + \rho \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq ra_m \leq a_{m-1} \leq \ldots \leq a_i \leq a_0 \) then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{a_0 + 2|a| - 2r(a_0 + |a|)}.
\]

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results.

2. MAIN RESULTS

Theorem 2.1: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( k \geq 1, \xi \geq 0, a_m \neq 0, a_i - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq k a_m \leq a_{m-1} \leq \ldots \leq a_1 \leq a_0 \) and for some \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq tb_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2(k|a_m + a_n| + t(|b_m + b_n|) - |a_m| - |b_m| + \xi + \eta) + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.
\]

Corollary 2.2: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( k \geq 1, \xi \geq 0, a_m \neq 0, 0 \leq a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq \ldots \geq a_1 \geq a_0 > 0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, 0 < b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq tb_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 > 0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2[(2k - 1)a_m + (2t - 1)b_m + \xi + \eta] - (a_0 + b_0)}.
\]

Corollary 2.3: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( k \geq 1, \xi \geq 0, a_m \neq 0, a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq \ldots \geq a_1 \geq a_0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq tb_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2(k|a_m + a_n| + b_m - |a_m| + \xi) + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.
\]

Corollary 2.4: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( k \geq 1, \xi \geq 0, a_m \neq 0, a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq \ldots \geq a_1 \geq a_0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq tb_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2(a_m + t(|b_m| + b_n) - |b_m| + \eta) + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.
\]

Corollary 2.5: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( k \geq 1, \xi \geq 0, a_m \neq 0, a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq k a_m \geq a_{m-1} \geq \ldots \geq a_1 \geq a_0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq kb_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2(k|a_m + a_n + b_m| - |a_m| - |b_m| + \eta) + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.
\]

Corollary 2.6: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq a_m \geq a_{m-1} \leq \ldots \leq a_1 \geq a_0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq b_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2(a_m + b_m) + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.
\]

Corollary 2.7: Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) with \( Re(\alpha_i) = a_i \) and \( Im(\alpha_i) = b_i \) such that for some \( 0 < a_n - \xi \leq a_{n-1} \leq \ldots \leq a_{m+1} \leq a_m \geq a_{m-1} \leq \ldots \leq a_1 \geq a_0 \) and \( t \geq 1, \eta \geq 0, b_m \neq 0, b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq b_m \geq b_{m-1} \leq \ldots \geq b_1 \geq b_0 > 0 \), then all the zeros of \( P(z) \) does not vanish in the disk
\[
|z| < \frac{|a_0|}{2[a_m - b_m] - (a_0 + b_0)}.
\]

Remark 2.8: By taking \( a_i > 0 \) and \( b_i > 0 \) for \( i = 0,1,2,\ldots,n \) in Theorem 2.1, it reduces to Corollary 2.2.

Remark 2.9: By taking \( \eta = 0 \) and \( t = 1 \) in Theorem 2.1, it reduces to Corollary 2.3.

Remark 2.10: By taking \( \xi = 0 \) and \( k = 1 \) in Theorem 2.1, it reduces to Corollary 2.4
Remark 2.11: By taking \( \eta = \xi \) and \( t = k \) in Theorem 2.1, it reduces to Corollary 2.5.

Remark 2.12: By taking \( \eta = 0 \) and \( k = t = 1 \) in Theorem 2.1, it reduces to Corollary 2.6.

Remark 2.13: By taking \( \eta = 0, k = t = 1 \) and \( a_i > 0, b_i > 0 \) for \( i = 0, 1, 2, \ldots, n \), in Theorem 2.1, it reduces to Corollary 2.7.

Remark 2.14: By taking \( b_i = 0 \) and \( \rho = \rho \) in Theorem 1, it reduces to Theorem 1.5.

Theorem 2.15: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( 0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_n \neq 0, a_n + \xi \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq \ldots \leq a_k \leq k a_0 \) and for some \( 0 < \mu \leq 1, t \geq 1, \eta \geq 0, b_m \neq 0, b_n + \eta \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{k(|a_0| + |a_0| + \tau(|b_0| + b_0) + X + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|))}.
\]

where \( X = 2(|a_m| + |b_m| + \xi + \eta - \tau(|a_m| + a_m) - \mu(|b_m| + b_m)) \).

Corollary 2.16: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( 0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_n \neq 0, a_n + \xi \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq \ldots \leq a_k \leq k a_0 \) and for some \( 0 < \mu \leq 1, t \geq 1, \eta \geq 0, b_m \neq 0, b_n + \eta \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{2(k a_0 + t b_0 - \tau a_m - \mu b_m + a_n + b_n + \xi + \eta) + (a_m + b_m - a_0 - b_0)}.
\]

Corollary 2.17: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( 0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_n \neq 0, a_n + \xi \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq \ldots \leq a_k \leq k a_0 \) and for some \( b_n \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{2(|b_m| + \eta - \mu(|b_m| + b_m)) + |a_n| + |b_n| + a_n + b_n - |a_0|}.
\]

Corollary 2.18: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( a_n \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq a_m \leq a_{m-1} \leq \ldots \leq a_k \) and for some \( 0 < \mu \leq 1, t \geq 1, \eta \geq 0, b_m \neq 0, b_n + \eta \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{2(|b_m| + \eta - \mu(|b_m| + b_m)) + |a_n| + |b_n| + a_n + b_n - |a_0|}.
\]

Corollary 2.19: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( 0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_n \neq 0, a_n + \xi \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq \ldots \leq a_k \) and for some \( b_n \neq 0, b_n + \xi \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{k(|a_0| + |b_0| + a_0 + b_0) + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|))}.
\]

Corollary 2.20: Let \( P(z) = \sum_{i=0}^{n} a_{i} z^{i} \) be a polynomial of degree \( n \) with \( \text{Re}(a_i) = a_i \) and \( \text{Im}(a_i) = b_i \) such that for some \( a_n \geq a_{n-1} \geq \ldots \geq a_{m+1} \geq a_m \leq a_{m-1} \leq \ldots \leq a_k \) and for some \( b_n \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \ldots \leq b_t \leq t b_0 \), then all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{(a_0 + b_0) - (a_n + b_n) + |a_n| + |b_n| + 2(a_n + b_n)}.
\]
Remark 2.22: By taking $a_i > 0$ and $b_i > 0$ for $i = 0, 1, 2, \ldots, n$, in Theorem 2, it reduces to Corollary 2.16.

Remark 2.23: By taking $\eta = 0$ and $t = \mu = 1$ in Theorem 2.15, it reduces to Corollary 2.17.

Remark 2.24: By taking $\xi = 0$ and $k = \tau = 1$ in Theorem 2.15, it reduces to Corollary 2.18.

Remark 2.25: By taking $\eta = \xi, \mu = \tau$ and $t = k$ in Theorem 2.15, it reduces to Corollary 2.19.

Remark 2.26: By taking $\eta = \xi = 0$ and $\mu = \tau = k = t = 1$ in Theorem 2.15, it reduces to Corollary 2.20.

Remark 2.27: By taking $\eta = \xi = 0$ and $\mu = \tau = k = t = 1$ and $a_i > 0, b_i > 0$ for $i = 0, 1, 2, \ldots, n$, in Theorem 2.15, it reduces to Corollary 2.21.

Remark 2.28: By taking $b_i = 0$, $\tau = r$ and $\xi = \rho$ in Theorem 2.15, it reduces to Theorem 1.6.

Theorem 2.29: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n$ with $Re(a_i) = a_i$ and $Im(a_i) = b_i$ such that for some $k \geq 1$, $\xi \geq 0$, $a_m \neq 0$, $a_m - \xi \leq a_{m-1} \leq \ldots \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq \ldots \geq a_1 \geq a_0$ and for some $0 < \mu \leq 1$, $\tau \geq 1$, $\eta \geq 0$, $b_n \neq 0$, $b_n + \eta \geq b_{n-1} \geq \ldots \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq \ldots \leq b_1 \leq t b_0$, then all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{t(|b_0| + b_0) + (a_0 + |b_0|) + X_2 + |a_n| + |b_n| - a_n + b_n},$$

where $X_2 = 2 k (|a_m| + a_m - |b_m| + |b_n| - \mu(|b_m| + b_n) + \xi + \eta)$.

Theorem 2.30: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n$ with $Re(a_i) = a_i$ and $Im(a_i) = b_i$ such that for some $0 < \tau \leq 1$, $k \geq 1$, $\xi \geq 0$, $a_m \neq 0$, $a_m + \xi \geq a_{m-1} \geq \ldots \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq \ldots \leq a_1 \leq \kappa a_0$ and for some $\eta \geq 0$, $b_n \neq 0$, $b_n - \eta \leq b_{n-1} \leq \ldots \leq b_{m+1} \leq \tau b_m \geq b_{m-1} \geq \ldots \geq b_1 \geq b_0$, then all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{k(|a_m| + a_m - |b_m| + b_n) + X_3 + |a_n| - b_n + |a_n| + |b_n|},$$

where $X_3 = 2 [ |a_m| - |b_m| - \mu |a_m| + a_m + t(|b_m| + b_n) + \xi + \eta]$.

3. Proofs of the Theorems

Proof of the Theorem 2.1:

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{m+1} z^{m+1} + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$

And $R(z) = (z-1)J(z)$ so that

Then $R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + \cdots + a_n z + a_n)$

$$= a_0 z^{n+1} - (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m} + (a_m - a_{m+1}) z^{n-m} + \cdots + a_n z + a_n$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < f$ for $i = 0, 1, 2, \ldots, n-1$. Now

$$|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^n + |a_m - a_{m+1}||z|^n + \cdots + |a_n||z|^{n-m} + |a_{m-1} - a_m||z|^{n-m} + \cdots + |a_n||z|^{n-m} \}$$

$$+ \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^n + |a_m - a_{m+1}||z|^n + \cdots + |a_n||z|^{n-m} + |a_{m-1} - a_m||z|^{n-m} + \cdots + |a_n||z|^{n-m} \}$$
Proof of the Theorem 2.30:

This shows that all the zeros of \( R(z) \) whose modulus is greater than 1 lie in the closed disk
\[
|z| > \frac{1}{\alpha_0} \left( \frac{2(1-a_0)+2(2k-1)}{2(1-a_0)+2(2k-1)} \right).
\]

Therefore, it follows that all the zeros of \( R(z) \) and hence \( J(z) \) lie in
\[
|z| < \frac{1}{\alpha_0} \left( \frac{2(1-a_0)+2(2k-1)}{2(1-a_0)+2(2k-1)} \right).
\]

Since \( P(z) = z^n J(z) \), it follows by replacing \( z \) by \( \frac{1}{z} \),
\[
|z| \geq \frac{1}{|\alpha_0|} \left( \frac{2(1-a_0)+2(2k-1)}{2(1-a_0)+2(2k-1)} \right).
\]

Hence \( P(z) \) does not vanish in the disk
\[
|z| < \frac{1}{|\alpha_0|} \left( \frac{2(1-a_0)+2(2k-1)}{2(1-a_0)+2(2k-1)} \right).
\]

This completes the proof of the Theorem 2.1.

**Proof of the Theorem 2.15:** Proof of the Theorem 2.15 is similar to that the proof of Theorem 2.1.

**Proof of the Theorem 2.29:** Proof of the Theorem 2.29 is similar to that the proof of Theorem 2.1.

**Proof of the Theorem 2.30:** Proof of the Theorem 2.30 is similar to that the proof of Theorem 2.1.

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Source of support: Nil, Conflict of interest: None Declared.

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