On gα-Separation Axioms

S. Balasubramanian*1 and M. Lakshmi Sarada2

1Department of Mathematics, Government Arts College (A), Karur – 639 005, Tamilnadu (INDIA)

2Department of Mathematics, A.M.G. Degree College, Chilakaluripet – 522 616, Andhrapradesh (INDIA)

E-mail: mani55682@rediffmail.com1 and lakhsa77492@yahoo.com2

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ABSTRACT

In this paper by using gα-open sets we define almost gα-normality and mild gα-normality also we continue the study of further properties of gα-normality. We show that these three axioms are regular open hereditary. We also define the class of almost gα-irresolute mappings and show that gα-normality is invariant under almost gα-irresolute M- gα-open continuous surjection.

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1. Introduction:

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the T1 and T2 spaces, namely, S1 and S2. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vyjayanthi studied v-Normal Almost-v-Normal, Mildly-v-Normal, gα-US, gα-S1 and gα-S2. Also we examine gα-convergence, sequentially gα-compact, sequentially gα-continuous maps, and sequentially sub gα-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. Preliminaries:

Definition 2.1: A X is called
(i) g-closed if cl A ⊆ U whenever A ⊆ U and U is open in X.
(ii) gα-closed if αcl(A) ⊆ U whenever A ⊆ U and U is open in X.

Definition 2.2: A function f is said to be almost–pre-irresolute if for each x in X and each pre-neighborhood V of f(x), pcl(f−1(V)) is a pre-neighborhood of x.

Definition 2.3: A space X is said to be
(i) T1 (T2) if for any x ≠ y in X, there exist (disjoint) open sets U; V in X such that x∈ U and y∈ V.
(ii) weakly Hausdorff if each point of X is the intersection of regular closed sets of X.
(iii) normal[resp: mildly normal] if for any pair of disjoint [resp: regular-closed]closed sets F1 and F2, there exist disjoint open sets U and V such that F1 ⊂ U and F2 ⊂ V.

*Corresponding author: S. Balasubramanian*1, E-mail: mani55682@rediffmail.com

(iv) almost normal if for each closed set A and each regular closed set B such that \( A \cap B = \emptyset \), there exist disjoint open sets U and V such that \( A \subseteq U \) and \( B \subseteq V \).

(v) weakly regular if for each pair consisting of a regular closed set A and a point \( x \) such that \( A \cap \{ x \} = \emptyset \), there exist disjoint open sets U and V such that \( x \in U \) and \( A \subseteq V \).

(vi) A subset A of a space X is S-closed relative to X if every cover of A by semiopen sets of X has a finite subfamily whose closures cover A.

(vii) \( R_0 \) if for any point x and a closed set F with \( x \notin F \) in X, there exists a open set G containing F but not x.

(viii) \( R_1 \) if for \( x, y \in X \) with \( \text{cl}\{x\} \neq \text{cl}\{y\} \), there exist disjoint open sets U and V such that \( \text{cl}\{x\} \subseteq U \), \( \text{cl}\{y\} \subseteq V \).

(ix) US-space if every convergent sequence has exactly one limit point to which it converges.

(x) pre-US space if every pre-convergent sequence has exactly one limit point to which it converges.

(xi) pre-\( S_1 \) if it is pre-US and every sequence \( \langle x_n \rangle \) pre-converges with subsequence of \( \langle x_n \rangle \) pre-side points.

(xii) pre-\( S_2 \) if it is pre-US and every sequence \( \langle x_n \rangle \) in X pre-converges which has no pre-side point.

(xiii) is weakly countable compact if every infinite subset of X has a limit point in X.

(xiv) Baire space if for any countable collection of closed sets with empty interior in X, their union also has empty interior in X.

**Definition 2.4:**

Let \( A \subseteq X \). Then a point \( x \) is said to be a

(i) limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( x \neq y \).

(ii) \( T_0 \)-limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( \text{cl}\{x\} \neq \text{cl}\{y\} \), or equivalently, such that they are topologically distinct.

(iii) pre-\( T_0 \)-limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( p\text{cl}\{x\} \neq p\text{cl}\{y\} \), or equivalently, such that they are topologically distinct.

**Note 1:** Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the \( T_0 \)-axiom is precisely to ensure that any two distinct points are topologically distinct.

**Example 1:** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\{a\}, \{b, c\}, \{a, b, c\}, X, \emptyset\} \). Then \( b \) and \( c \) are the limit points but not the \( T_0 \)-limit points of the set \( \{b, c\} \). Further \( d \) is a \( T_0 \)-limit point of \( \{b, c\} \).

**Example 2:** Let \( X = (0, 1) \) and \( \tau = \{\emptyset, X, \text{and } U_n = (0, 1-1/n), n = 2, 3, 4, \ldots \} \). Then every point of \( X \) is a limit point of \( X \). Every point of \( X \setminus U_2 \) is a \( T_0 \)-limit point of \( X \), but no point of \( U_2 \) is a \( T_0 \)-limit point of \( X \).

**Definition 2.5:** A set \( A \) together with all its \( T_0 \)-limit points will be denoted by \( T_0 \)-cl\( A \).

**Note 2:**

(i) Every \( T_0 \)-limit point of a set \( A \) is a limit point of the set but the converse is not true in general.

(ii) In \( T_0 \)-space both are same.

**Note 3:** \( R_0 \)-axiom is weaker than \( T_1 \)-axiom. It is independent of the \( T_0 \)-axiom. However \( T_1 = R_0 + T_0 \)

**Note 4:** Every countable compact space is weakly countable compact but converse is not true in general. However, a \( T_1 \)-space is weakly countable compact iff it is countable compact.

### 3. \( g\alpha \)-\( T_0 \) LIMIT POINT:

**Definition 3.01:** In \( X \), a point \( x \) is said to be a \( g\alpha \)-\( T_0 \)-limit point of \( A \) if each \( g\alpha \)-open set containing \( x \) contains some point \( y \) of \( A \) such that \( g\alpha\text{cl}\{x\} \neq g\alpha\text{cl}\{y\} \), or equivalently; such that they are topologically distinct with respect to \( g\alpha \)-open sets.
Example 3: regular open set ⇒ open set ⇒ α-open set ⇒ gα-open set we have r-T0-limit point ⇒ T0-limit point ⇒ α-T0-limit point ⇒ gα-T0-limit point

Definition 3.02: A set A together with all its gα-T0-limit points is denoted by T0gαcl(A)

Lemma 3.01: If x is a gα-T0-limit point of a set A then x is gα-limit point of A.

Lemma 3.02: If X is gα-T0-space then every gα-T0-limit point and every gα-limit point are equivalent.

Corollary 3.03: If X is r-T0-space then every gα-T0-limit point and every gα-limit point are equivalent.

Theorem 3.04: For x ∉ A,
   (i) x is a gα-T0-limit point of A if and only if x ∉ gαcl(A).
   (ii) x is not a gα-T0-limit point of A if and only if x ∉ gαcl(A).
   (iii) x is a gα-T0-limit point of A if and only if x ∈ gαcl(A).

Corollary 3.05: In a regular T0-space, a point x is a gα-limit point of A if and only if x ∈ A.

Lemma 3.06: In X, if x is a gα-limit point of a set A, then in each of the following cases x becomes gα-T0-limit point of A
   (i) gαcl{x} = {x}.
   (ii) A - {x} is gα-open.

Corollary 3.07: In X, if x is a limit point of a set A, then in each of the following cases x becomes gα-T0-limit point of A
   (i) gαcl{x} = {x}.
   (ii) A - {x} is gα-open.

4. gα-T0 AND gα-R0 AXIOMS, i = 0, 1:

In view of Lemma 3.06 (ii), gα-R0-axiom implies the equivalence of the concept of limit point of a set with that of gα-T0-limit point of a set. The converse, however, is not true in general. But for the converse, if x is a gα-T0-limit point of a set A, then x ∈ gαcl(A). If x is a limit point of a set A, then x ∈ αcl(A).

Lemma 4.01: In a space X, a point x is a gα-T0-limit point of A if and only if x ∈ gαcl(A).
Since every r-\(R_\alpha\)-space is \(g\alpha-R_\alpha\)-space, we have the following corollary

**Corollary 4.04:** The following conditions are equivalent:
(i) \(X\) is a r-\(R_\alpha\) space
(ii) For any \(x, y\) in \(X\), if \(x \in g\alpha\{y\}\), then \(x\) is not a \(g\alpha-T_\alpha\)-limit point of \([y]\)
(iii) A point \(g\alpha\)-closure set has no \(g\alpha-T_\alpha\)-limit point in \(X\)
(iv) A singleton set has no \(g\alpha-T_\alpha\)-limit point in \(X\).

**Theorem 4.05:** In a \(g\alpha-R_\alpha\) space \(X\), a point \(x\) is \(g\alpha-T_\alpha\)-limit point of \(A\) iff every \(g\alpha\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.

If \(g\alpha-R_\alpha\) space is replaced by r\(R_\alpha\) space in the above theorem, we have the following corollaries:

**Corollary 4.06:** In an r\(R_\alpha\)-space \(X\),
(i) If a point \(x\) is r\(T_\alpha\)-limit point of a set then every \(g\alpha\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.
(ii) If a point \(x\) is \(g\alpha-T_\alpha\)-limit point of a set then every \(g\alpha\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.

**Theorem 4.07:** \(X\) is \(g\alpha-R_\alpha\) space iff a set \(A\) of the form \(A = \bigcup g\alpha\{x_{i,1} \to n\}\) a finite union of point closure sets has no \(g\alpha-T_\alpha\)-limit point.

**Corollary 4.08:** If \(X\) is \(R_\alpha\) space and
(i) If \(A = \bigcup g\alpha\{x_{i,1} \to n\}\) a finite union of point closure sets has no \(g\alpha-T_\alpha\)-limit point.
(ii) If \(X = \bigcup g\alpha\{x_{i,1} \to n\}\) then \(X\) has no \(g\alpha-T_\alpha\)-limit point.

**Theorem 4.09:** The following conditions are equivalent:
(i) \(X\) is \(g\alpha-R_\alpha\)-space
(ii) For any \(x\) and a set in \(X\), \(x\) is a \(g\alpha-T_\alpha\)-limit point of \(A\) iff every \(g\alpha\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.

Various characteristic properties of \(g\alpha-T_\alpha\)-limit points studied so far is enlisted in the following theorem for a ready reference.

**Theorem 4.10:** In a \(g\alpha-R_\alpha\)-space, we have the following:
(i) A singleton set has no \(g\alpha-T_\alpha\)-limit point in \(X\).
(ii) A finite set has no \(g\alpha-T_\alpha\)-limit point in \(X\).
(iii) A point \(g\alpha\)-closure has no set \(g\alpha-T_\alpha\)-limit point in \(X\).
(iv) A finite union point \(g\alpha\)-closure sets have no set \(g\alpha-T_\alpha\)-limit point in \(X\).
(v) For \(x, y\) in \(X\), if \(x \in g\alpha\{y\}\) then \(x = y\).
(vi) For any \(x, y\) in \(X\), if \(x \neq y\) iff either \(x\) is \(g\alpha-T_\alpha\)-limit point of \([y]\) nor \(y\) is \(g\alpha-T_\alpha\)-limit point of \([x]\).
(vii) For any \(x, y\) in \(X\), if \(x \neq y\) iff \(g\alpha\{x\} \cap g\alpha\{y\} = \emptyset\).
(viii) Any point \(x\) in \(X\) is a \(g\alpha-T_\alpha\)-limit point of a set \(A\) in \(X\) iff every \(g\alpha\)-open set containing \(x\) contains infinitely many points of \(A\) with each of which \(x\) is topologically distinct.

**Theorem 4.11:** \(X\) is \(g\alpha-R_\alpha\) iff for any \(g\alpha\)-open set \(U\) in \(X\) and points \(x, y\) such that \(x \in X - U, y \in U\), there exists a \(g\alpha\)-open set \(V\) in \(X\) such that \(y \in V \cup U, x \in V\).

**Lemma 4.12:** In \(g\alpha-R_\alpha\) space \(X\), if \(x\) is a \(g\alpha-T_\alpha\)-limit point of \(X\), then for any non empty \(g\alpha\)-open set \(U\), there exists a non empty \(g\alpha\)-open set \(V\) such that \(V \subset U, x \notin g\alpha\{V\}\).

**Lemma 4.13:** In a \(g\alpha\)-regular space \(X\), if \(x\) is a \(g\alpha-T_\alpha\)-limit point of \(X\), then for any non empty \(g\alpha\)-open set \(U\), there exists a non empty \(g\alpha\)-open set \(V\) such that \(g\alpha\{V\} \subset U, x \notin g\alpha\{V\}\).

**Corollary 4.14:** In a regular space \(X\),
(i) if \(x\) is a \(g\alpha-T_\alpha\)-limit point of \(X\), then for any non empty \(g\alpha\)-open set \(U\), there exists a non empty \(g\alpha\)-open set \(V\) such that \(g\alpha\{V\} \subset U, x \notin g\alpha\{V\}\).
(ii) if \(x\) is a \(T_\alpha\)-limit point of \(X\), then for any non empty \(g\alpha\)-open set \(U\), there exists a non empty \(g\alpha\)-open set \(V\) such that \(g\alpha\{V\} \subset U, x \notin g\alpha\{V\}\).
**Theorem 4.15:** If $X$ is a $g\alpha$-compact $g\alpha$-$R_1$-space, then $X$ is a Baire Space.

**Proof:** Let $\{A_\alpha\}$ be a countable collection of $g\alpha$-closed sets of $X$, each $A_\alpha$ having empty interior in $X$. Take $A_1$, since $A_1$ has empty interior, $A_1$ does not contain any $g\alpha$-open set say $U_0$. Therefore we can choose a point $y \in U_0$ such that $y \notin A_1$. For $X$ is $g\alpha$-regular, and $y \in (X-A_1) \cap U_0$, a $g\alpha$-open set, we can find a $g\alpha$-open set $U_1$ in $X$ such that $y \in U_1$, $g\alpha h(U_1) \subset (X-A_1) \cap U_0$. Hence $U_1$ is a non empty $g\alpha$-open set in $X$ such that $g\alpha h(U_1) \subset U_0$ and $g\alpha h(U_1) \cap A_1 = \emptyset$. Continuing this process, in general, for given non empty $g\alpha$-open set $U_{n-1}$, we can choose a point of $U_{n-1}$ which is not in the $g\alpha$-closed set $A_n$ and a $g\alpha$-open set $U_n$ containing this point such that $g\alpha h(U_n) \subset U_{n-1}$ and $g\alpha h(U_n) \cap A_n = \emptyset$. Thus we get a sequence of nested non empty $g\alpha$-closed sets which satisfies the finite intersection property. Therefore $\cap g\alpha h(U_n) \neq \emptyset$. Then some $x \in \cap g\alpha h(U_n)$ which in turn implies that $x \in U_{n-1}$ as $g\alpha h(U_n) \subset U_{n-1}$ and $x \notin A_n$ for each $n$.

**Corollary 4.16:** If $X$ is a compact $g\alpha$-$R_1$-space, then $X$ is a Baire Space.

**Corollary 4.17:** Let $X$ be a $g\alpha$-compact $g\alpha$-$R_1$-space. If $\{A_\alpha\}$ is a countable collection of $g\alpha$-closed sets in $X$, each $A_\alpha$ having non-empty $g\alpha$-interior in $X$, then there is a point of $X$ which is not in any of the $A_\alpha$.

**Corollary 4.18:** Let $X$ be a $g\alpha$-compact $R_1$-space. If $\{A_\alpha\}$ is a countable collection of $g\alpha$-closed sets in $X$, each $A_\alpha$ having non-empty $g\alpha$-interior in $X$, then there is a point of $X$ which is not in any of the $A_\alpha$.

**Theorem 4.19:** Let $X$ be a non empty compact $g\alpha$-$R_1$-space. If every point of $X$ is a $g\alpha$-$T_\theta$-limit point of $X$ then $X$ is uncountable.

**Proof:** Since $X$ is non empty and every point is a $g\alpha$-$T_\theta$-limit point of $X$, $X$ must be infinite. If $X$ is countable, we construct a sequence of $g\alpha$-open sets $\{V_\alpha\}$ in $X$ as follows:

Let $X = V_1$, then for $x_0$, is a $g\alpha$-$T_\theta$-limit point of $X$, we can choose a non empty $g\alpha$-open set $V_2$ in $X$ such that $V_2 \subset V_1$ and $x_0 \notin g\alpha h(V_2)$. Next for $x_0$, and non empty $g\alpha$-open set $V_2$, we can choose a non empty $g\alpha$-open set $V_3$ in $X$ such that $V_3 \subset V_2$ and $x_0 \notin g\alpha h(V_3)$. Continuing this process for each $x_0$, and a non empty $g\alpha$-open set $V_n$, we can choose a non empty $g\alpha$-open set $V_{n+1}$ in $X$ such that $V_{n+1} \subset V_n$ and $x_0 \notin g\alpha h(V_{n+1})$.

Now consider the nested sequence of $g\alpha$-closed sets $g\alpha h(V_1) \supset g\alpha h(V_2) \supset g\alpha h(V_1) \supset \ldots \supset g\alpha h(V_n) \supset \ldots$

Since $X$ is $g\alpha$-compact and $\{g\alpha h(V_n)\}$ the sequence of $g\alpha$-closed sets satisfies finite intersection property. By Cantor’s intersection theorem, there exists an $x$ in $X$ such that $x \in g\alpha h(V_n)$. Further $x \in X$ and $x \in V_n$, which is not equal to any of the points of $X$. Hence $X$ is uncountable.

**Corollary 4.20:** Let $X$ be a non empty $g\alpha$-compact $g\alpha$-$R_1$-space. If every point of $X$ is a $g\alpha$-$T_\theta$-limit point of $X$ then $X$ is uncountable.

5. $g\alpha$–$T_\theta$-IDENTIFICATION SPACES AND $g\alpha$–SEPARATION AXIOMS:

**Definition 5.01:** Let $(X, \tau)$ be a topological space and let $\mathcal{R}$ be the equivalence relation on $X$ defined by $x \equiv y$ iff $g\alpha h(x) = g\alpha h(y)$

**Problem 5.02:** show that $x \equiv y$ iff $g\alpha h(x) = g\alpha h(y)$ is an equivalence relation

**Definition 5.03:** The space $(X_0, Q(X_0))$ is called the $g\alpha$-$T_\theta$–identification space of $(X, \tau)$, where $X_0$ is the set of equivalence classes of $\mathcal{R}$ and $Q(X_0)$ is the decomposition topology on $X_0$.

Let $P_X: (X, \tau) \rightarrow (X_0, Q(X_0))$ denote the natural map

**Lemma 5.04:** If $x \in X$ and $A \subseteq X$, then $x \in g\alpha h(A)$ iff every $g\alpha$-open set containing $x$ intersects $A$.

**Theorem 5.05:** The natural map $P_X: (X, \tau) \rightarrow (X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O \in PO(X, \tau)$ and $(X_0, Q(X_0))$ is $g\alpha$-$T_0$

**Proof:** Let $O \in PO(X, \tau)$ and let $C \subseteq P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then $g\alpha h(y) = g\alpha h(x)$, which, by lemma, implies $y \in O$. Since $\tau \subset PO(X, \tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \subseteq \tau$, which implies $P_X$ is closed and open.
Let \( G, \Phi X_0 \) such that \( G \neq H \) and let \( x \in G \) and let \( y \in H \). Then \( g \alpha \mathcal{l}(x) \neq g \alpha \mathcal{l}(y) \), which implies that \( x \notin g \alpha \mathcal{l}(y) \) or \( y \notin g \alpha \mathcal{l}(x) \). Since \( P_X \) is continuous and open, then \( G \in \Phi X = P_X(X - g \alpha \mathcal{l}(y)) \notin PO(X_0, Q(X_0)) \) and \( H \notin \Phi A 

**Theorem 5.06:** The following results are equivalent:

(i) \( X \) is \( g \alpha R_0 \) (ii) \( X_0 = \{ g \alpha \mathcal{l}(x): x \in X \} \) and (iii) \( (\Phi X_0, Q(X_0)) \) is \( g \alpha T_1 \)

**Proof:**

(i) \( \Rightarrow \) (ii) Let \( C \in \Phi X_0 \) and let \( x \in C \). If \( y \in C \), then \( y \notin g \alpha \mathcal{l}(x) = g \alpha \mathcal{l}(x) \), which implies \( C \notin g \alpha \mathcal{l}(x) \). If \( y \notin g \alpha \mathcal{l}(x) \), then \( x \notin g \alpha \mathcal{l}(y) \). Hence, if \( y \notin g \alpha \mathcal{l}(x) \), then \( x \notin g \alpha \mathcal{l}(y) \), which implies \( g \alpha \mathcal{l}(y) = g \alpha \mathcal{l}(x) \) and \( y \in C \). Hence \( X_0 = \{ g \alpha \mathcal{l}(x): x \in X \} \)

(ii) \( \Rightarrow \) (iii) Let \( A \neq B \in X_0 \). Then there exists \( x, y \in X \) such that \( A = g \alpha \mathcal{l}(x) \); \( B = g \alpha \mathcal{l}(y) \), and \( g \alpha \mathcal{l}(x) \cap g \alpha \mathcal{l}(y) = \phi \). Then \( A \in C = P_X(X - g \alpha \mathcal{l}(y)) \neq PO(X_0, Q(X_0)) \) and \( B \notin C \). Hence, \( X_0 = \{ g \alpha \mathcal{l}(x): x \in X \} \)

(iii) \( \Rightarrow \) (i) Let \( x \in g \alpha \mathcal{l}(X) \). Let \( y \in U \) and \( C_x, C_y \in X_0 \) containing \( x \) and \( y \) respectively. Then \( y \in g \alpha \mathcal{l}(X) \), which implies \( C_x \neq C_y \) and there exists \( g \alpha \)-open set \( A \) such that \( C_x \in A \) and \( C_y \notin A \). Since \( P_X \) is continuous and open, then \( y \notin B = P_X^{-1}(A) \notin g \alpha \mathcal{l}(X) \) and \( x \in B 

**Theorem 5.07:** \( (X, \tau) = g \alpha R_1 \) iff \( (\Phi X_0, Q(X_0)) = g \alpha T_2 \)

The proof is straightforward from using theorems 5.05 and 5.06 and is omitted.

**Theorem 5.08:** \( X \) is \( g \alpha T_i \); \( i = 0, 1, 2 \) iff there exists a \( g \alpha \)-continuous, almost–open, 1–1 function from \( (X, \tau) \) into a \( g \alpha T_i \) space; \( i = 0, 1, 2 \) respectively.

**Proof:** If \( X \) is \( g \alpha T_i \); \( i = 0, 1, 2 \), then the identity function on \( X \) satisfies the desired properties. The converse is (ii) part of Theorem 2.13.

The following example shows that if \( f: (X, \tau) \rightarrow (Y, \sigma) \) is continuous, \( g \alpha \)-open, bijective, \( A \in PO(Y, \sigma) \), and \( (Y, \sigma) = g \alpha T_i \); \( i = 0, 1, 2 \), then \( f^{-1}(A) \) need not be \( g \alpha \)-open and \( (X, \tau) \) need not be \( g \alpha T_i \); \( i = 0, 1, 2 \)

**Theorem 5.09:** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is \( g \alpha \)-continuous, \( g \alpha \)-open, and \( x \in X \) such that \( g \alpha \mathcal{l}(x) = g \alpha \mathcal{l}(y) \), then \( g \alpha \mathcal{l}(f(x)) = g \alpha \mathcal{l}(f(y)) \).

**Theorem 5.10:** The following are equivalent

(i) \( (X, \tau) \) is \( g \alpha T_0 \)

(ii) Elements of \( X_0 \) are singleton sets and

(iii) There exists a \( g \alpha \)-continuous, \( g \alpha \)-open, 1–1 function \( f: (X, \tau) \rightarrow (Y, \sigma) \), where \( (Y, \sigma) = g \alpha T_0 \)

**Proof:** (i) is equivalent to (ii) and (i) \( \Rightarrow \) (iii) are straightforward and is omitted.

(iii) \( \Rightarrow \) (i) Let \( x, y \in X \) such that \( f(x) \neq f(y) \), which implies \( g \alpha \mathcal{l}(f(x)) \neq g \alpha \mathcal{l}(f(y)) \). Then by theorem 5.09 \( g \alpha \mathcal{l}(x) \neq g \alpha \mathcal{l}(y) \). Hence \( (X, \tau) \) is \( g \alpha T_0 \)

**Corollary 5.11:** A space \( (X, \tau) \) is \( g \alpha T_i \); \( i = 1, 2 \) iff \( (X, \tau) \) is \( g \alpha T_{i-1} \); \( i = 1, 2 \), respectively, and there exists a \( g \alpha \)-continuous, \( g \alpha \)-open, 1–1 function \( f: (X, \tau) \rightarrow \) into a \( g \alpha T_i \) space.

**Definition 5.04:** \( f: X \rightarrow Y \) is point– \( g \alpha \)-closure 1–1 iff for \( x, y \in X \) such that \( g \alpha \mathcal{l}(x) \neq g \alpha \mathcal{l}(y) \), \( g \alpha \mathcal{l}(f(x)) \neq g \alpha \mathcal{l}(f(y)) \).

**Theorem 5.12:**

(i) \( f: (X, \tau) \rightarrow (Y, \sigma) \) is point– \( g \alpha \)-closure 1–1 and \( (X, \tau) \) is \( g \alpha T_0 \) then \( f \) is 1–1

(ii) \( f: (X, \tau) \rightarrow (Y, \sigma) \), where \( (X, \tau) \) and \( (Y, \sigma) \) are \( g \alpha T_0 \) then \( f \) is point– \( g \alpha \)-closure 1–1 iff \( f \) is 1–1

**Proof:** omitted

The following result can be obtained by combining results for \( g \alpha T_i \) identification spaces, \( g \alpha \)-induced functions and \( g \alpha T_i \) spaces; \( i = 1, 2 \).
Theorem 5.13: X is gα-R₀; i = 0,1 iff there exists a gα-continuous, almost-open point–gα-closure 1-1 function f: (X, τ) into a gα-R₁ space; i = 0,1 respectively.

6. gα-Normal; Almost gα-normal and Mildly gα-normal spaces:

Definition 6.1: A space X is said to be gα-normal if for any pair of disjoint closed sets F₁ and F₂, there exist disjoint gα-open sets U and V such that F₁ ⊂ U and F₂ ⊂ V.

Example 4: Let X = {a, b, c} and τ = {∅, {a}, {b, c}, X}. Then X is gα-normal.

Example 5: Let X = {a, b, c, d} and τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. Then X is not gα-normal and is not normal. We have the following characterization of gα-normality.

Theorem 6.1: For a space X the following are equivalent:
(i) X is gα-normal.
(ii) For every pair of open sets U and V whose union is X, there exist gα-closed sets A and B such that A⊂U, B⊂V and A∪B = X.
(iii) For every closed set F and every open set G containing F, there exists a gα-open set U such that F⊂U⊂gαcl(U)⊂G.

Proof: (a)⇒(b): Let U and V be a pair of open sets in a gα-normal space X such that X = U∪V. Then X-U, X-V are disjoint closed sets. Since X is gα-normal there exist disjoint gα-open sets U₁ and V₁ such that X-U ⊂ U₁ and X-V ⊂ V₁. Let A = X-U₁, B = X – V₁. Then A and B are gα-closed sets such that A⊂U, B⊂V and A∪B = X.

(b) ⇒(c): Let F be a closed set and G be an open set containing F. Then F∩G is an open set whose union is X. Then by (b), there exist gα-closed sets W₁ and W₂ such that W₁ ⊂ F∩G and W₂ ⊂ G and W₁∪W₂ = X. Then F⊂ X-W₁, X-G ⊂ X-W₂ and (X-W₁)∩(X-W₂) = ∅. Let U= X-W₁ and V= X-W₂. Then U and V are disjoint gα-open sets such that F⊂U⊂X-V⊂gαcl(U)⊂G.

(c) ⇒ (a): Let F₁ and F₂ be any two disjoint closed sets of X. Put G = X-F₂, then F₁∩G = ∅, F₁⊂G where G is an open set. Then by (c), there exists a gα-open set U of X such that F₁⊂U⊂gαcl(U)⊂G. It follows that F₁⊂X-gαcl(U) = V, say, then U is gα-open and U∩V = ∅. Hence F₁ and F₂ are separated by gα-open sets U and V. Therefore X is gα-normal.

Theorem 6.2: A regular open subspace of a gα-normal space is gα-normal.

Proof: Let Y be a regular open subspace of a gα-normal space X. Let A and B be disjoint closed subsets of Y. As Y is regular open, A,B are closed sets of X. By gα-normality of X, there exist disjoint gα-open sets U and V in X such that A⊂U and B⊂V, U∩Y and V∩Y are gα-open in Y such that A⊂U∩Y and B⊂V∩Y. Hence Y is gα-normal.

Example 6: Let X = {a, b, c} with τ = {∅, {a}, {b}, {a, b}, X} is gα-normal and gα-regular.

However we observe that every gα-normal gα-R₀ space is gα-regular.

Now, we define the following.

Definition 6.2: A function f: X → Y is said to be almost – gα-irresolute if for each x in X and each gα-neighborhood V of f(x), gαcl(f⁻¹(V)) is a gα-neighborhood of x.

Clearly every gα-irresolute map is almost gα-irresolute.

The Proof of the following lemma is straightforward and hence omitted.

Lemma 6.1: f is almost gα-irresolute iff f⁻¹(V) ⊂ gα-Int(gαcl(f⁻¹(V))) for every V∈gαO(Y). Now we prove the following.

Lemma 6.2: f is almost gα-irresolute iff f(gαcl(U)) ⊂ gαcl(f(U)) for every U∈gαO(X).

Proof: Let U∈gαO(X). Suppose y∉gαcl(f(U)). Then there exists V∈gαO(Y) such that V∩f(U) = ∅. Hence f⁻¹(V)∩U = ∅. Since U∈gαO(X), we have gα-Int(gαcl(f⁻¹(V))) ⊂ gαcl(U) = ∅. Then by lemma 6.1, f⁻¹(V)∩gαcl(U) = ∅ and hence V∩gαcl(f(U)) = ∅. This implies that y∉f(gαcl(U)).
Conversely, if \( V \subseteq g\alpha O(Y) \), then \( W = X - g\alpha l(f(I(V))) \subseteq g\alpha O(X) \). By hypothesis, \( f(g\alpha l(f(W))) \subseteq g\alpha l(f(W))) \) and hence \( X - g\alpha \text{-int}(g\alpha l(f(I(V)))) = g\alpha l(W) \subseteq f(I(V)) \subseteq f(g\alpha l(f(X - f(I(V))))) \subseteq f^{-1}[g\alpha l(Y - V)] = f^{-1}(Y - V) = X - f(I(V)) \).

Therefore, \( f(I(V)) \subseteq g\alpha \text{-int}(g\alpha l(f(I(V)))) \). By lemma 6.1, \( f \) is almost \( g\alpha \)-irresolute.

Now we prove the following result on the invariance of \( g\alpha \)-normality.

**Theorem 6.3:** If \( f \) is an \( M \)- \( g\alpha \)-open continuous almost \( g\alpha \)-irresolute function from a \( g\alpha \)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( g\alpha \)-normal.

**Proof:** Let \( A \) be a closed subset of \( Y \) and \( B \) be an open set containing \( A \). Then by continuity of \( f \), \( f^{-1}(A) \) is closed and \( f^{-1}(B) \) is an open set of \( X \) such that \( f^{-1}(A) \subseteq f^{-1}(B) \). As \( X \) is \( g\alpha \)-normal, there exists a \( g\alpha \)-open set \( U \) in \( X \) such that \( f^{-1}(A) \subseteq U \subseteq g\alpha l(U) \subseteq f^{-1}(B) \). Then \( f^{-1}(A) \subseteq f(U) \subseteq f(g\alpha l(U)) \subseteq f(f^{-1}(B)) \). Since \( f \) is \( M \)-\( g\alpha \)-open almost \( g\alpha \)-irresolute surjection, we obtain \( A \subseteq f(U) \subseteq g\alpha l(f(U)) \subseteq B \). Then again by Theorem 6.1 the space \( Y \) is \( g\alpha \)-normal.

**Lemma 6.3:** A mapping \( f \) is \( M \)- \( g\alpha \)-closed if and only if for each subset \( B \) in \( Y \) and for each \( g\alpha \)-open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a \( g\alpha \)-open set \( V \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

Now we prove the following:

**Theorem 6.4:** If \( f \) is an \( M \)- \( g\alpha \)-closed continuous function from a \( g\alpha \)-normal space onto a space \( Y \), then \( Y \) is \( g\alpha \)-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 [9] and lemma 6.3, we prove that the following result.

**Theorem 6.5:** If \( f \) is an \( M \)- \( g\alpha \)-closed map from a weakly Hausdorff \( g\alpha \)-normal space \( X \) onto a space \( Y \) such that \( f(I(y)) \) is \( S \)-closed relative to \( X \) for each \( y \in Y \), then \( Y \) is \( g\alpha \)-\( T_2 \).

**Proof:** Let \( y_1 \) and \( y_2 \) be any two distinct points of \( Y \). Since \( X \) is weakly Hausdorff, \( f^{-1}(y_1) \) and \( f^{-1}(y_2) \) are disjoint closed subsets of \( X \) by lemma 2.2 [9]. As \( X \) is \( g\alpha \)-normal, there exist disjoint \( g\alpha \)-open sets \( V_1 \) and \( V_2 \) such that \( f^{-1}(y_i) \subseteq V_i \) for \( i = 1, 2 \). Since \( f \) is \( M \)- \( g\alpha \)-closed, there exist \( g\alpha \)-open sets \( U_1 \) and \( U_2 \) containing \( y_1 \) and \( y_2 \) such that \( f^{-1}(U_i) \subseteq V_i \) for \( i = 1, 2 \). Then it follows that \( U_1 \cap U_2 = \phi \). Hence \( Y \) is \( g\alpha \)-\( T_2 \).

**Theorem 6.6:** For a space \( X \) we have the following:

(a) If \( X \) is normal then for any disjoint closed sets \( A \) and \( B \), there exist disjoint \( g\alpha \)-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \);

(b) If \( X \) is normal then for any closed set \( A \) and any open set \( V \) containing \( A \), there exists an \( g\alpha \)-open set \( U \) of \( X \) such that \( A \subseteq U \subseteq g\alpha l(U) \subseteq V \).

**Definition 6.2:** \( X \) is said to be almost \( g\alpha \)-normal if for each closed set \( A \) and each regular closed set \( B \) such that \( A \cap B = \phi \), there exist disjoint \( g\alpha \)-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

Clearly, every \( g\alpha \)-normal space is almost \( g\alpha \)-normal, but not conversely in general.

**Example 7:** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} \). Then \( X \) is almost \( g\alpha \)-normal and not \( g\alpha \)-normal.

Now, we have characterization of almost \( g\alpha \)-normality in the following.

**Theorem 6.7:** For a space \( X \) the following statements are equivalent:

(i) \( X \) is almost \( g\alpha \)-normal

(ii) For every pair of sets \( U \) and \( V \), one of which is open and the other is regular open whose union is \( X \), there exist \( g\alpha \)-closed sets \( G \) and \( H \) such that \( G \subseteq U \), \( H \subseteq V \) and \( G \cup H = X \).

(iii) For every closed set \( A \) and every regular open set \( B \) containing \( A \), there is a \( g\alpha \)-open set \( V \) such that \( A \subseteq V \subseteq g\alpha l(V) \subseteq B \).

**Proof:** (a) \( \Rightarrow \) (b) Let \( U \) be an open set and \( V \) be a regular open set in an almost \( g\alpha \)-normal space \( X \) such that \( U \cup V = X \). Then \( (X - U) \) is closed and \( (X - V) \) is regular closed set with \( (X - U) \cap (X - V) = \phi \). By almost \( g\alpha \)-normality of \( X \), there
exist disjoint $g\alpha$-open sets $U_1$ and $V_1$ such that $X-U \subset U_1$ and $X-V \subset V_1$. Let $G = X-U_1$ and $H = X-V_1$. Then $G$ and $H$ are $g\alpha$-closed sets such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious.

One can prove that almost $g\alpha$-normality is also regular open hereditary.

Almost $g\alpha$-normality does not imply almost $g\alpha$-regularity in general. However, we observe that every almost $g\alpha$-normal $g\alpha$-$R_0$ space is almost $g\alpha$-regular.

Next, we prove the following.

**Theorem 6.8:** Every almost regular, $v$-compact space $X$ is almost $g\alpha$-normal.

Recall that a function $f : X \to Y$ is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost $g\alpha$-normality in the following.

**Theorem 6.9:** If $f$ is continuous $M$-$g\alpha$-open rc-continuous and almost $g\alpha$-irresolute surjection from an almost $g\alpha$-normal space $X$ onto a space $Y$, then $Y$ is almost $g\alpha$-normal.

**Definition 6.3:** A space $X$ is said to be mildly $g\alpha$-normal if for every pair of disjoint regular closed sets $F_1$ and $F_2$ of $X$, there exist disjoint $g\alpha$-open sets $U$ and $V$ such that $F_1 \subset U$ and $F_2 \subset V$.

**Example 8:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $X$ is mildly $g\alpha$-regular.

We have the following characterization of mild $g\alpha$-normality.

**Theorem 6.10:** For a space $X$ the following are equivalent.

(i) $X$ is mildly $g\alpha$-normal.

(ii) For every pair of regular open sets $U$ and $V$ whose union is $X$, there exist $g\alpha$-closed sets $G$ and $H$ such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(iii) For any regular closed set $A$ and every regular open set $B$ containing $A$, there exists a $g\alpha$-open set $U$ such that $A \subset U \subset g\alpha l(U) \subset B$.

(iv) For every pair of disjoint regular closed sets, there exist $g\alpha$-open sets $U$ and $V$ such that $A \subset U$, $B \subset V$ and $g\alpha l(U) \cap g\alpha l(V) = \emptyset$.

This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild $g\alpha$-normality is regular open hereditary.

We define the following.

**Definition 6.4:** A space $X$ is weakly $g\alpha$-regular if for each point $x$ and a regular open set $U$ containing $\{x\}$, there is a $g\alpha$-open set $V$ such that $x \in V \subset cl V \subset U$.

**Example 9:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $X$ is weakly $g\alpha$-regular.

**Example 10:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $X$ is not weakly $g\alpha$-regular.

**Theorem 6.11:** If $f : X \to Y$ is an $M$-$g\alpha$-open rc-continuous and almost $g\alpha$-irresolute function from a mildly $g\alpha$-normal space $X$ onto a space $Y$, then $Y$ is mildly $g\alpha$-normal.

**Proof:** Let $A$ be a regular closed set and $B$ be a regular open set containing $A$. Then by rc-continuity of $f$, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since $X$ is mildly $g\alpha$-normal, there exists a $g\alpha$-open set $V$ such that $f^{-1}(A) \subset V \subset g\alpha l(V) \subset f^{-1}(B)$ by Theorem 6.10. As $f$ is $M$-$g\alpha$-open and almost $g\alpha$-irresolute surjection, it follows that $f(V) \in g\alpha l(Y)$ and $A \subset f(V) \subset g\alpha l(f(V)) \subset B$. Hence $Y$ is mildly $g\alpha$-normal.
Theorem 6.12: If \( f: X \to Y \) is rc-continuous, M- \( g\alpha \)-closed map from a mildly \( g\alpha \)-normal space \( X \) onto a space \( Y \), then \( Y \) is mildly \( g\alpha \)-normal.

7. \( g\alpha \)-US spaces:

Definition 7.1: A sequence \( \langle x_n \rangle \) is said to be \( g\alpha \)-converges to a point \( x \) of \( X \), written as \( \langle x_n \rangle \to^{g\alpha} x \) if \( \langle x_n \rangle \) is eventually in every \( g\alpha \)-open set containing \( x \).

Clearly, if a sequence \( \langle x_n \rangle \) \( \alpha \)-converges to a point \( x \) of \( X \), then \( \langle x_n \rangle \) \( g\alpha \)-converges to \( x \).

Definition 7.2: \( X \) is said to be \( g\alpha \)-US if every sequence \( \langle x_n \rangle \) in \( X \) \( g\alpha \)-converges to a unique point.

Theorem 7.1: Every \( g\alpha \)-US space is \( g\alpha \)-T1.

Proof: Let \( X \) be a \( g\alpha \)-US space. Let \( x \) and \( y \) be two distinct points of \( X \). Consider the sequence \( \langle x_n \rangle \) where \( x_n = x \) for every \( n \). Clearly, \( \langle x_n \rangle \to^{g\alpha} x \). Also, since \( x \neq y \) and \( X \) is \( g\alpha \)-US, \( \langle x_n \rangle \) cannot \( g\alpha \)-converge to \( y \), i.e., there exists a \( g\alpha \)-open set \( V \) containing \( y \) but not \( x \). Similarly, for the sequence \( \langle y_n \rangle \) where \( y_n = y \) for all \( n \), and proceeding as above we get a \( g\alpha \)-open set \( U \) containing \( x \) but not \( y \). Thus, the space \( X \) is \( g\alpha \)-T1.

Theorem 7.2: Every \( g\alpha \)-T2 space is \( g\alpha \)-US.

Proof: Let \( X \) be \( g\alpha \)-T2 space and \( \langle x_n \rangle \) be a sequence in \( X \). If possible suppose that \( \langle x_n \rangle \) \( g\alpha \)-converge to two distinct points \( x \) and \( y \). That is, \( \langle x_n \rangle \) is eventually in every \( g\alpha \)-open set containing \( x \) and also in every \( g\alpha \)-open set containing \( y \). This is contradiction since \( X \) is \( g\alpha \)-T2 space. Hence the space \( X \) is \( g\alpha \)-US.

Definition 7.3: A set \( F \) is sequentially \( g\alpha \)-closed if every sequence in \( F \) \( g\alpha \)-converges to a point in \( F \).

Theorem 7.3: \( X \) is \( g\alpha \)-US iff the diagonal set in \( X \times X \) is sequentially \( g\alpha \)-closed subset of \( X \times X \).

Proof: Let \( X \) be \( g\alpha \)-US. Let \( \langle x_n, x_n \rangle \) be a sequence in \( \Delta \). Then \( \langle x_n, x_n \rangle \) is a sequence in \( X \). As \( X \) is \( g\alpha \)-US, \( \langle x_n, x_n \rangle \to^{g\alpha} x \) for a unique \( x \in X \), i.e., if \( \langle x_n, x_n \rangle \to^{g\alpha} y \) and \( y \). Thus, \( x = y \). Hence \( \Delta \) is sequentially \( g\alpha \)-closed.

Conversely, let \( \Delta \) be sequentially \( g\alpha \)-closed and let \( \langle x_n, x_n \rangle \to^{g\alpha} x \) and \( y \). Hence \( \langle x_n, x_n \rangle \to^{g\alpha} (x, y) \). Since \( \Delta \) is sequentially \( g\alpha \)-closed, \( (x, y) \in \Delta \) which means that \( x = y \) implies space \( X \) is \( g\alpha \)-US.

Definition 7.4: A subset \( G \) of a space \( X \) is said to be sequentially \( g\alpha \)-compact if every sequence in \( G \) has a subsequence which \( g\alpha \)-converges to a point in \( G \).

Theorem 7.4: In a \( g\alpha \)-US space every sequentially \( g\alpha \)-compact set is sequentially \( g\alpha \)-closed.

Proof: Let \( X \) be \( g\alpha \)-US space. Let \( Y \) be a sequentially \( g\alpha \)-compact subset of \( X \). Let \( \langle x_n \rangle \) be a sequence in \( Y \). Suppose that \( \langle x_n \rangle \) \( g\alpha \)-converges to a point in \( X \times Y \). Let \( \langle x_{n_p} \rangle \) be subsequence of \( \langle x_n \rangle \) that \( g\alpha \)-converges to a point \( y \in Y \). Since \( Y \) is sequentially \( g\alpha \)-compact. Also, let a subsequence \( \langle x_{n_p} \rangle \) of \( \langle x_n \rangle \) \( g\alpha \)-converge to \( x \in X \times Y \). Since \( \langle x_{n_p} \rangle \) is a sequence in the \( g\alpha \)-US space \( X \times Y \). \( x = y \). Thus, \( Y \) is sequentially \( g\alpha \)-closed set.

Next, we give a hereditary property of \( g\alpha \)-US spaces.

Theorem 7.5: Every regular open subset of a \( g\alpha \)-US space is \( g\alpha \)-US.

Proof: Let \( X \) be a \( g\alpha \)-US space and \( Y \subset X \) be an regular open set. Let \( \langle x_n \rangle \) be a sequence in \( Y \). Suppose that \( \langle x_n \rangle \) \( g\alpha \)-converges to \( x \) and \( y \) in \( Y \). We shall prove that \( \langle x_n \rangle \) \( g\alpha \)-converges to \( x \) and \( y \) in \( X \). Let \( U \) be any \( g\alpha \)-open subset of \( X \) containing \( x \) and \( V \) be any \( g\alpha \)-open set of \( X \) containing \( y \). Then, \( U \cap Y \) and \( V \cap Y \) are \( g\alpha \)-open sets in \( Y \). Therefore, \( \langle x_n \rangle \) is eventually in \( U \cap Y \) and \( V \cap Y \) and so in \( U \) and \( V \). Since \( X \) is \( g\alpha \)-US, this implies that \( x = y \). Hence the subspace \( Y \) is \( g\alpha \)-US.

Theorem 7.6: A space \( X \) is \( g\alpha \)-T2 iff it is both \( g\alpha \)-R1 and \( g\alpha \)-US.

Proof: Let \( X \) be \( g\alpha \)-T2 space. Then \( X \) is \( g\alpha \)-R1 and \( g\alpha \)-US by Theorem 7.2.

Conversely, let \( X \) be both \( g\alpha \)-R1 and \( g\alpha \)-US space. By Theorem 7.1, \( X \) is both \( g\alpha \)-T1 and \( g\alpha \)-R1 and, it follows that space \( X \) is \( g\alpha \)-T2.

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Definition 7.5: A point y is a $g\alpha$-cluster point of a sequence $<x_n>$ iff $<x_n>$ is frequently in every $g\alpha$-open set containing x. The set of all $g\alpha$-cluster points of $<x_n>$ will be denoted by $g\alpha$-cl($x_n$).

Definition 7.6: A point y is $g\alpha$-side point of a sequence $<x_n>$ if y is a $g\alpha$-cluster point of $<x_n>$ but no subsequence of $<x_n>$ $g\alpha$-converges to y.

Now, we define the following.

Definition 7.7: A space X is said to be
(i) $g\alpha$-S$_1$ if it is $g\alpha$-US and every sequence $<x_n>$ $g\alpha$-converges with subsequence of $<x_n>$ $g\alpha$-side points.
(ii) $g\alpha$-S$_2$ if it is $g\alpha$-US and every sequence $<x_n>$ in X $g\alpha$-converges which has no $g\alpha$-side point.

Lemma 7.1: Every $g\alpha$-S$_2$ space is $g\alpha$-S$_1$ and Every $g\alpha$-S$_1$ space is $g\alpha$-US.

Using sequentially continuous functions, we define sequentially $g\alpha$-continuous functions.

Definition 7.8: A function f is said to be sequentially $g\alpha$-continuous at $x \in X$ if $f(x_n) \rightarrow ^{g\alpha} f(x)$ whenever $<x_n> \rightarrow ^{g\alpha} x$.

If $f$ is sequentially $g\alpha$-continuous at all $x \in X$, then $f$ is said to be sequentially $g\alpha$-continuous.

Theorem 7.7: Let $f$ and $g$ be two sequentially $g\alpha$-continuous functions. If Y is $g\alpha$-US, then the set $A = \{ x | f(x) = g(x) \}$ is sequentially $g\alpha$-closed.

Proof: Let Y be $g\alpha$-US and suppose that there is a sequence $<x_n>$ in A $g\alpha$-converging to $x \in X$. Since $f$ and $g$ are sequentially $g\alpha$-continuous functions, $f(x_n) \rightarrow ^{g\alpha} f(x)$ and $g(x_n) \rightarrow ^{g\alpha} g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, A is sequentially $g\alpha$-closed.

Next, we prove the product theorem for $g\alpha$-US spaces.

Theorem 7.8: Product of arbitrary family of $g\alpha$-US spaces is $g\alpha$-US.

Proof: Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ where $X_\lambda$ is $g\alpha$-US. Let a sequence $<x_n>$ in X $g\alpha$-converges to $x$ (= $x_\lambda$) and $y$ (= $y_\lambda$). Then $<x_n>$ $\rightarrow ^{g\alpha} x_\lambda$ and $y_\lambda$ for all $\lambda \in \Lambda$. For suppose there exists a $\mu \in \Lambda$ such that $<x_n>$ does not $g\alpha$-converges to $x_\mu$. Then there exists a $\tau_{g\alpha}$- $g\alpha$-open set $U_\mu$ containing $x_\mu$ such that $<x_n>$ is not eventually in $U_\mu$. Consider the set $U = \prod_{\lambda \in \Lambda} X_\lambda \times U_\mu$. Then U is a $g\alpha$-open subset of X and $x \in U$. Also, $<x_n>$ is not eventually in U, which contradicts the fact that $<x_n>$ $\rightarrow ^{g\alpha} x$. Thus we get $<x_n>$ $\rightarrow ^{g\alpha} x_\lambda$ and $y_\lambda$ for all $\lambda \in \Lambda$. Since $X_\lambda$ is $g\alpha$-US for each $\lambda \in \Lambda$. Thus $x = y$. Hence X is $g\alpha$-US.

8. Sequentially sub- $g\alpha$-continuity:

Definition 8.1: A function f is said to be
(i) sequentially nearly $g\alpha$-continuous if for each point $x \in X$ and each sequence $<x_n>$ $\rightarrow ^{g\alpha} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ such that $f(x_{nk}) \rightarrow ^{g\alpha} f(x)$.

(ii) sequentially sub- $g\alpha$-continuous if for each point $x \in X$ and each sequence $<x_n>$ $\rightarrow ^{g\alpha} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ and a point $y \in Y$ such that $f(x_{nk})$ $\rightarrow ^{g\alpha} y$.

(iii) sequentially $g\alpha$-compact preserving if $f(K)$ is sequentially $g\alpha$-compact in Y for every sequentially $g\alpha$-compact set K of X.

Lemma 8.1: Every function f is sequentially sub- $g\alpha$-continuous if Y is a sequentially $g\alpha$-compact.

Proof: Let $<x_n>$ $\rightarrow ^{g\alpha} x$ in X. Since Y is sequentially $g\alpha$-compact, there exists a subsequence $\{f(x_{ak})\}$ of $\{f(x_n)\}$ $g\alpha$-converging to a point $y \in Y$. Hence f is sequentially sub- $g\alpha$-continuous.

Theorem 8.1: Every sequentially nearly $g\alpha$-continuous function is sequentially $g\alpha$-compact preserving.

Proof: Assume f is sequentially nearly $g\alpha$-continuous and K any sequentially $g\alpha$-compact subset of X. Let $<y_n>$ be any sequence in f (K). Then for each positive integer n, there exists a point $x_k \in K$ such that $f(x_k) = y_n$. Since $<x_n>$ is a sequence in the sequentially $g\alpha$-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ $g\alpha$-converging to a point $x \in K$. By hypothesis, f is sequentially nearly $g\alpha$-continuous and hence there exists a subsequence $<x_{nk}>$ of $<x_{nk}>$ such that
Proof: There exists a subsequence \( \langle x_\alpha > \) of \( \langle y_\alpha > \) \( g\alpha \)-converging to \( f(x) \in f(K) \). This shows that \( f(K) \) is sequentially \( g\alpha \)-compact set in \( Y \).

**Theorem 8.2:** Every sequentially \( \alpha \)-continuous function is sequentially \( g\alpha \)-continuous.

**Proof:** Let \( f \) be a sequentially \( \alpha \)-continuous and \( \langle x_\alpha > \rightarrow ^\alpha x \in X \). Then \( \langle x_\alpha > \rightarrow ^\alpha x \). Since \( f \) is sequentially \( \alpha \)-continuous, \( f(x_\alpha ) \rightarrow f(x) \). But we know that \( \langle x_\alpha > \rightarrow ^\alpha x \) implies \( \langle x_\alpha > \rightarrow ^g\alpha x \) and hence \( f(x_\alpha ) \rightarrow ^g\alpha f(x) \) implies \( f \) is sequentially \( g\alpha \)-continuous.

**Theorem 8.3:** Every sequentially \( g\alpha \)-compact preserving function is sequentially sub- \( g\alpha \)-continuous.

**Proof:** Suppose \( f \) is a sequentially \( g\alpha \)-compact preserving function. Let \( x \) be any point of \( X \) and \( \langle x_\alpha > \) any sequence in \( X \) \( g\alpha \)-converging to \( x \). We shall denote the set \( \{ x_n | n = 1, 2, 3 \ldots \} \) by \( A \) and \( K = A \cup \{ x \} \). Then \( K \) is sequentially \( g\alpha \)-compact since \( \langle x_\alpha > \rightarrow ^\alpha x \). By hypothesis, \( f \) is sequentially \( g\alpha \)-compact preserving and hence \( f(K) \) is a sequentially \( g\alpha \)-compact set of \( Y \). Since \( \{ f(x_n) \} \) is a sequence in \( f(K) \), there exists a subsequence \( \{ f(x_{n_k}) \} \) of \( \{ f(x_\alpha ) \} \) \( g\alpha \)-converging to a point \( y \in f(K) \). This implies that \( f \) is sequentially sub- \( g\alpha \)-continuous.

**Theorem 8.4:** A function \( f: X \rightarrow Y \) is sequentially \( g\alpha \)-compact preserving iff \( f_{/K}: K \rightarrow f(K) \) is sequentially sub- \( g\alpha \)-continuous for each sequentially \( g\alpha \)-compact subset \( K \) of \( X \).

**Proof:** Suppose \( f \) is a sequentially \( g\alpha \)-compact preserving function. Then \( f(K) \) is sequentially \( g\alpha \)-compact set in \( Y \) for each sequentially \( g\alpha \)-compact set \( K \) of \( X \). Therefore, by Lemma 8.1 above, \( f_{/K}: K \rightarrow f(K) \) is sequentially \( g\alpha \)-continuous function.

Conversely, let \( K \) be any sequentially \( g\alpha \)-compact set of \( X \). Let \( \langle y_\alpha > \) be any sequence in \( f(K) \). Then for each positive integer \( n \), there exists a point \( x_n \in K \) such that \( f(x_n) = y_\alpha \). Since \( \langle x_\alpha > \) is a sequence in the sequentially \( g\alpha \)-compact set \( K \), there exists a subsequence \( \langle x_{n_k} > \) of \( \langle x_\alpha > \) \( g\alpha \)-converging to a point \( x \in K \). By hypothesis, \( f_{/K}: K \rightarrow f(K) \) is sequentially sub- \( g\alpha \)-continuous and hence there exists a subsequence \( \langle y_{n_k} > \) of \( \langle y_\alpha > \) \( g\alpha \)-converging to a point \( y \in f(K) \). This implies that \( f(K) \) is sequentially \( g\alpha \)-compact set in \( Y \). Thus, \( f \) is sequentially \( g\alpha \)-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub- \( g\alpha \)-continuous function to be sequentially \( g\alpha \)-compact preserving.

**Corollary 8.1:** If \( f \) is sequentially sub- \( g\alpha \)-continuous and \( f(K) \) is sequentially \( g\alpha \)-closed set in \( Y \) for each sequentially \( g\alpha \)-compact set \( K \) of \( X \), then \( f \) is sequentially \( g\alpha \)-compact preserving function.

**Proof:** Omitted.

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**References:**


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