THE DEFICIENT DISCRETE QUARTIC SPLINE INTERPOLATION

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ABSTRACT
In this paper, we have obtained a precise error estimate of the deficient discrete Quartic spline interpolate matching the given function values at the mesh point and its difference at mid points and also boundary points.

Key word's: Error Bounds, Interpolation, Deficient Discrete Quartic splines.

1. INTRODUCTION:
Deficient splines are more useful than usual splines as they require less continuity requirement at mesh points. Discrete splines have been introduced by Mangasarian and Schumaker [7] in connection with certain studies of minimization problems involving difference. Rana and Dubey [8] have obtains asymptotically precise estimate of the difference between discrete cubic spline interpolant and the function interpolated, which is sometimes used to smooth a histogram. In the Direction of some constructive aspect's of discrete splines, we refer to Astor and Duris [1], Jia [4], Dikshit and Powar [3] and Rana [9]. The object of the present paper is to study the existence, uniqueness and convergence properties of deficient discrete quartic spline matching the given functional values at mesh points and its differences at mid points and also boundary point.

Let us consider a mesh \( P \) on \([0,1]\), which is defined by

\[
0 = x_0 < x_1 < ... < x_n = 1.
\]

such that \( x_i - x_{i-1} = P_i \), for \( i = 1,2,...,n \). and \( P = \max_{i\in\mathbb{N}} P_i \). Throughout, \( h \) will be represent a given positive real number. Consider a real continuous function \( S(x,h) \) defined over \([0,1]\) which is such that its restriction \( S_i \) on \([x_{i-1},x_i]\) is a polynomial of degree 4 or less for \( i = 1,2,...,n \), then \( S(x,h) \) defines a discrete quartic spline if

\[
D_h^{(j)} S_i(x,h) = D_h^{(j)} S_{i+1}(x,h) \quad j = 0,1,2,3. 
\]

(1.1)

where the difference operator \( D_h \) are defined as

\[
D_h^{(0)} f(x) = f(x), \quad D_h^{(1)} f(x) = (f(x+h) - f(x-h))/2h
\]

\[
D_h^{(2)} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.
\]

2. EXISTENCE AND UNIQUENESS

The class of deficient discrete quartic splines is denoted by \( S(4,2,\Delta,h) \) where \( S^*(4,2,\Delta,h) \) denotes the class of all discrete deficient quartic splines which satisfies the boundary condition

\[
D_h^{(1)} S(x_0,h) = D_h^{(1)} f(x_0),
\]

\[
D_h^{(1)} S(x_n,h) = D_h^{(1)} f(x_n),
\]

(2.1)
We introduce the following interpolating conditions

\[
S(x_i) = f(x_i) \quad i = 0, \ldots, n-1, n
\]

\[
D_h^{(1)} S(Z_i) = D_h^{(1)} f(Z_i) \quad i = 0, \ldots, n-1.
\]  

(2.2)

where \( Z_i = \frac{x_i + x_{i-1}}{2} \)

Infact, we shall prove following:

**Theorem: 2.1** For any \( h > 0 \), there exists a unique deficient discrete quartic spline \( S(x, h) \in S^*(4,2,P,h) \) which satisfies the conditions (2.1) and (2.2).

Let \( E(Z) \) be a quartic spline polynomial defined on [0,1]. It can be easily verified that

\[
E(Z) = E(0)P_1(Z) + D_h^{(1)} E'(1/2)P_2(Z) + E(1)P_3(Z) + D_h^{(1)} E'(0)P_4(Z) + D_h^{(1)} E'(1)P_5(Z)
\]  

(2.4)

where \( P_1(Z) = \frac{(1-z)^2}{2} (2 + 4z - z^2) \)

\[ P_2(Z) = -z^2 (1 - Z)^2 \]

\[ P_3(Z) = \frac{z^2}{2} (7 + 6z - z^2) \]

\[ P_4(Z) = z (1 - z)^2 (1 + z/4) \]

\[ P_5(Z) = \frac{z^2 (z-1) (3+z)}{4} \]

Let \( S(x,h) \) be a discrete quartic polynomial \([x_i, x_{i+1}]\) and \( \frac{x-x_i}{P_i} = Z \) then we can write

\[
S_i(x,h) = f(x_i)P_1(z) + p_i D_h^{(1)} f(z_{i+1}) P_2(z) + f(x_{i+1})P_3(z) + p_i D_h^{(1)} f(x_i) P_4(z) + p_i D_h^{(1)} f(x_{i+1}) P_5(z)
\]  

(2.5)

which is clearly satisfy the condition (2.1) and (2.2).

Let \( S_i(a,b) = ap_i^2 + bh_i^2 \), where \( a,b \) are real numbers and \( m_i = D_h^{(1)} S_i(x,h) \), we apply the continuity of the second difference of \( S_i(x,h) \) at \( x_i \) in (2.5) to see that

\[
p_i^3 m_{i-1} S_{i-1}(5,1) + (p_i^3 S_{i-1}(9,1) + p_i^{1+} S_{i-1}(7,-1)) m_i + p_i^3 m_{i+1} S_{i+1}(3,-1) = \frac{p_i^3}{p_i} F_i + \frac{p_i^3}{p_{i-1}} F_i^*
\]  

(2.6)

where \( F_i = (f_i - f_{i+1}) S_{i-1}(10,0) + p_i f_i^{[1]}(Z_i) S_{i-1}(4,4) \)

We can easily see that excess of the absolute value of the coefficient of \( m_i \) over the sum of absolute value of the coefficients of \( m_{i-1} \) and \( m_{i+1} \) in (2.6) under the condition of theorem 2.1 is given by

\[ G_i(h) = [p_i^3 S_{i-1}(4,0) + p_i^{3+} S_{i-1}(4,0)] \]

Thus the coefficient matrix of the system of equations (2.6) is diagonally dominant and hence invertible, therefore the system of equation (2.6) has unique solution, which completes the proof of the theorem 2.1.

**Remark:** In the case, when \( h \to 0 \) theorem 2.1 gives the corresponding results for continuous \( C^2 \) quartic spline interpolation under condition (2.1) and (2.2).

3. ERROR BOUNDS:

Now system of equation (2.6) may be written as

\[ A(h) \cdot M(h) = F. \]
where $A(h)$ is coefficient matrix and $M(h) = m_i(h)$. However, as already shown in the proof of theorem 2.1 $A(h)$ is invertible. Denoting the inverse of $A(h)$ by $A^{-1}(h)$ we note that row max norm $A^{-1}(h)$ satisfies the following inequality

$$\| A^{-1}(h) \| \leq y(h).$$

(3.1)

where $y(h) = \max \{ l_i(h) \}^{-1}$ for convenience, we assume in this section that $l = Nh$ when $N$ is positive integer. It is also assumed that the mesh points $[x_i]$ are such that

$$x_i \in [0,1], \text{ for } i = 0,1,\ldots,n.$$

where discrete interval $[0,1]$ is the set of points $\{0,N,\ldots,Nh\}$ for a function $f$ and two distinct points $x_1, x_2$ in it’s domain, the first divided difference is defined by

$$[x_1,x_2]f = \frac{f(x_1) - f(x_2)}{(x_1 - x_2)}$$

and

$$[x_1,x_2,x_3]f = \frac{[x_2,x_3]f - [x_1,x_2]f}{(x_3 - x_1)}.$$  

(3.2)

For convenience, we write $f^{(1)}$ for $D_h^{(1)}f$, $f_i^{(1)}$ for $D_h^{(1)}f_i$ and $w(f,p)$ for modules of continuity of $f$, the discrete norm of a function $f$ over the interval $[0,1]$ is defined by

$$\| f \| = \max_{x \in [0,1]} |f(x)|$$

(3.3)

We shall obtain in the following the bound of error functions $e(x) = s(x,h) - f(x)$ over the discrete interval $[0,1]$.  

**Theorem: 3.1** Suppose $S(x,h)$ is the discrete quartic splines interpolant of theorem 2.1 then

$$\| e(x) \| \leq k(p,h)w(f^{(1)},p)$$

(3.4)

and

$$\| e_i^{(1)}(x) \| \leq y(h)k^+(p,h)w(f^{(1)},p)$$

(3.5)

$$\| e_i^{(2)}(x) \| \leq k^{(+)}(p,h)w(f^{(1)},p)$$

(3.6)

where $k(p,h), k^+(p,h)$ and $k^{(+)}(p,h)$ are some positive functions of $p$ and $h$.

**Proof:** Writing $f(x_i) = f_i$. Equation (3.1) may be written as

$$A(h) \cdot e^{(1)}(x_i) = (f_i^{(1)})A(h) = (L_i) \text{ (say)}$$

(3.7)

when we replace $m_i(h)$ by

$$e_i^{(1)}(x_i) = s^{(1)}(x_i,h) - f_i^{(1)}$$

we need following lemma due to Lyche [5,6], to obtain inequity (3.4).

**Lemma: 3.1** Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be given sequence of non-negative real numbers such that $\sum a_i = \sum b_j$ then for any real valued function $f$, defined on discrete interval $[0,1]$, we have

$$\left| \sum_{i=1}^{m} a_i \left[ x_{i_1}, x_{i_2}, \ldots, x_{i_k} \right] f - \sum_{j=1}^{n} b_j \left[ y_{j_1}, y_{j_2}, \ldots, y_{j_k} \right] f \right| \leq W(f^{(k)},1-kh) \frac{\sum a_i}{k!}$$

(3.8)

where $x_{i_1}, y_{j_k} \in [0,1]$ for relevant values of $i, j$ and $k$. 

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We can write the equation (2.6) is of the form of error function as follows.

\[
[p_i^3 e_i^{[1]} S_{i-1}(5,1) + p_i^3 S_{i-1}(9,1) + p_i^3 S_i(7,-1)] e_i^{[1]} + p_i^3 e_i^{[1]} S_i(3,-1) = \frac{p_i^j}{p_i} F_i + \frac{p_i^j}{p_i} F_i^* \]

\[-[p_i^3 f_i^{[1]} S_{i-1}(5,1) + p_i^3 S_{i-1}(9,1) + p_i^3 S_i(7,-1)] f_i^{[1]} + p_i^3 f_i^{[1]} S_i(3,-1) = R_i(f) \quad \text{(say)} \quad (3.9)
\]

Writing equation (3.9) is of the form of divided difference and using Lemma 3.1, given by Lyche [5,6].

Since

\[
\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j
\]

We get

\[
l(R_i(f)) = \sum_{i=1}^{3} a_i [x_{i0}, x_{i1}]_j - \sum_{j=1}^{4} b_j [y_{j0}, y_{j1}]_j
\]

\[
\leq w(f^{[1]}, p) \sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j
\]

where

\[
a_1 = p_i^3 (10 p_i^2 - 2h^2) = p_i^3 S_{i-1}(10,-2),
\]

\[
a_2 = p_i^3 (14 p_i^2 + 2h^2) = p_i^3 S_i(14,2),
\]

\[
a_3 = 4 p_i^3 (h^2 + p_i^2) = p_i^3 S_{i-1}(4,4),
\]

\[
b_1 = p_i^3 (5 p_i^2 + h^2) = p_i^3 S_{i-1}(5,1),
\]

\[
b_2 = p_i^3 S_{i-1}(9,1) + p_i^3 S_i(7,-1)
\]

\[
b_3 = p_i^3 S_i(3,1) = p_i^3 S_{i-1}(3,-1)
\]

\[
b_4 = p_i^3 S_i(4,4) = p_i^3 S_{i-1}(4,4)
\]

and

\[
x_{10} = x_{i-1}, x_{11} = x_i = x_{20} = y_{i0} = y_{20} = y_{30} = y_{41}
\]

\[
x_{21} = x_{i21} = y_{21} = x_{31}, \ y_{31} = \alpha_{i+1}, \ y_{41} = \beta_{i+1}
\]

\[
x_{30} = z_{i+1}, y_{40} = z_i
\]

Now using the equations (3.9) and (3.10).

\[
|| e^{[1]}(x) || \leq y(h) k (p, h) w(f^{[1]}, p) \quad (3.11)
\]

\[
|| e^{[1]}(x) || \leq k_1 (p, h) w(f^{[1]}, p)
\]

This complete proof of inequality (3.5) of Theorem 3.1.

Next, we have to get \( e(x) \). Writing equation (2.5) in form of error function as follows.

\[
e(x) = p_i e_i^{[1]} P_i(t) + p_i e_i^{[1]} P_i(t) + L_i(f)
\]

where

\[
L_i(f) = f_i P_i(t) + p_i f_i^{[1]} (z_{i+1}) P_2(t) + f_i P_3(t) + p_i f_i^{[1]} P_4(t) + p_i f_i^{[1]} P_5(t) - f(x) \quad (3.12)
\]

We write \( L_i(f) \) in form of divided difference and using Lemma 3.1, we get

\[
| L_i(f) | \leq w(f^{[1]}, p) \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j
\]

\[
= p_i (z + z^2 - 2z^3 + z^4)
\]

\[
= p_i k(z, h) \quad \text{(say)} \quad (3.13)
\]
where

\[ a_1 = \frac{1}{2}(7z^2 + 6z^3 - z^4) \]
\[ a_2 = \frac{1}{4}(4z + 2z^3 - 7z^2 + z^4) \]
\[ a_3 = \frac{1}{4}(-3z^2 + 2z^3 + z^4) \]
\[ b_1 = z \]
\[ b_2 = z^2 + z - 2z^3 \]

and

\[ x_{10} = x_{20} = x_{30} = x_i \]
\[ x_{11} = x_{21} = x_{31} = x_{i+1} \]
\[ y_{10} = y_{i} \, , \, y_{11} = x_i \]
\[ y_{20} = z_{i+1}, \quad y_{21} = x_{i+1} \]

From equations (3.11) and (3.12) gives inequality (3.4) of theorem (3.1).

This complete proof of inequality of theorem 3.1.

Again we have to find error bound of second difference as follows.

\[ e^{[2]}(x) = [S_i^{[2]}(x) - f_i^{[2]}(x)] \]

Now

\[
P_i^2 e^{[2]}(x) = [f_i A_i^{[2]}(t) + P_i f^{(1)}(z_{i+1}) A_{i+1}^{[2]}(t) + P_i f^{(1)} A_i^{[2]}(t) + P_i e^{(1)} A_i^{[2]}(t) + e^{(1)} A_{i+1}^{[2]}(t)]
+ P_i [f_i^{(1)} A_i^{[2]}(t) + f_i^{(1)} A_{i+1}^{[2]}(t)] - P_i^2 f_i^{[2]}(x)
\]

(3.14)

For convinience we write (3.14) equation

\[
P_i^2 e^{[2]}(x) = P_i [f_i^{(1)} A_i^{[2]}(t) + e^{(1)} A_i^{[2]}(t)] + Q_i(f)
\]

(3.15)

where

\[
Q_i(f) = [f_i A_i^{[2]}(t) + P_i f(z_{i+1}) A_{i+1}^{[2]}(t) + f_i A_i^{[2]}(t) + P_i e^{(1)} A_i^{[2]}(t) + e^{(1)} A_{i+1}^{[2]}(t)] - P_i^2 f_i^{[2]}(x)
\]

We write \(Q_i(f)\) in form of divided difference and using Lemma 3.1, we get

\[
\|Q_i(f)\| = \sum_{i=1}^5 a_i [x_{i0}, x_{i1}] f - \sum_{j=1}^1 b_j [y_{j0}, y_{j1}] f
\leq \left(\sum_{i=1}^5 a_i = \sum_{j=1}^1 b_j\right) w(f^{(1)}, P)
\]

(3.16)

where

\[ a_1 = P_i [7 - 18z + (6z^2 + h^2)] \]
\[ a_2 = P_i [2 + 12z - 2(6z^2 + h^2)] \]
\[ a_3 = P_i [-7/2 + 3z + 1/2(6z^2 + h^2)] \]
\[ a_4 = P_i [-3/2 + 3z + 1/2(6z^2 + h^2)] \]
\[ a_5 = P_i \]
\[ b_1 = P_i \]
Since \( a_1 + a_2 + a_3 + a_4 + a_5 = P_i = b_i \)

i.e. \( \sum_{i=1}^{5} a_i = b_i = P_i \)

and

\[
\begin{align*}
  x_{10} & = y_{10} = x_{40} = x_i, \\
  x_{11} & = x_{i+1} = x_{31} = x_{41} = x_{51} \\
  x_{20} & = x_{i+1}, \quad x_{30} = x_{i-1}, \quad x_{31} = x_i \\
  x_{50} & = x = y_{11},
\end{align*}
\]

Hence

\[
|Q_i(f)| \leq P_i w(f^{(1)}, P)
\]  

(3.17)

By using (3.5) and (3.15) in (3.14) we get inequality (3.6) of theorem (3.1).

This complete proof of theorem 3.1.

**REFERENCE:**


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