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MENGER SPACE ON WEAK COMPATIBILITY AND FIXED POINT THEOREM

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ABSTRACT

In this paper we prove fixed point theorem for six self maps inmenger space.

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***Key words**: Menger space, t-norm common fixed points, compatible map, weak-compatible maps.*

1. INTRODUCTION

There have been a number of generalizations of metric space. Menger[6] one such generalizations Menger space initiated . It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [9]studied this concept and gave some fundamental results on this space. It is observed by many authors that contraction condition in metric space may be exactly translated into PM-space endowed with min norms. Sehgal and Bharucha-Reid [10] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a mile stone in developing fixed-point theorems in Menger space. Sessa [8] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [3] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [7]. Recently, Jungck and Rhoades [5] (also Dhage [1]) termed a pair of self maps to becoincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. This concept is most general among all the commutativity concepts in this field as every pair of R-weakly commuting self maps is compatible and each pair of compatible self maps is weakly compatible but the reverse is not true always. In this paper a fixed point theorem for six self maps has been proved using the concept of weak compatibility and compatibility of pair of self maps, which turns out to be a material generalization of the results of Mishra [7] and others. For the sake of completeness, following Mishra [7] and Sehgal and Bharucha Reid [10], we recall some definitions and known results in Menger space.

2. PRELIMINARIES

Defintion1.A mapping F: $R \to R+$ is called a distribution if it is non-decreasing left continuous with $\inf\{F(t): t \in R\} = 0$ and $\sup\{F(t): t \in R\} = 1$.We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0. \end{cases}$$

Definition 2: [7] A probabilistic metric space (PM-space) is an ordered pair (X, F) where X is an abstract set of elements and $F: X \times X \to L$ is defined by $(p, q) \to Fp, q$ where L is the set of all distribution functions, i.e., $L = \{Fp ,q : p, q \in X\}$, where the functions Fp, q satisfy:

(a) $Fp, q(x) = 1$ for all $x > 0$, if and only if $p = q$;

(b) $Fp, q\ (0) = 0$;

(c) $Fp, q = Fq, p$;

(d) If $Fp, q(x) = 1$ and $Fq, r\ (y) = 1$ then $Fp, r\ (x + y) = 1$.

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Definition 3: A mapping t*:* $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *t*-norm if

(e) $t(a, 1) = a$, $t(0, 0) = 0$;

(f) $t(a, b) = t(b, a)$;

(g) $t(c, d) \geq t(a, b)$ for $c \geq a$, $d \geq b$;

(h) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 4: A Menger space is a triplet (X, F, t), where (X,F) is PM-space and t is a t-norm such that for all $p, q, r \in X$ and for all $x, y \geq 0$

$Fp, r(x + y) \geq t\{Fp, q(x), Fq, r(y)\}$

Proposition 1: [10] *If (X, d) is a metric then the metric d induces a mapping $X \times X \rightarrow L$, defined by $Fp, q(x) = H(x - d(p, q))$, $p, q \in X$ and $x \in R$. Further, if the t-norm t: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X,F, t) is a Menger space. It is complete if (X, d) is complete.* The space (X, F, t) so obtained is called the induced Menger space.

Definition 5: [7] A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \rightarrow p$) if for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that $Fpn, p(\varepsilon) > 1 - \lambda$, for all $n \geq M(\varepsilon,\lambda)$. The sequence is said to be Cauchy if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon,\lambda)$ such that $Fp_n, p_m(\varepsilon)$ _ $1-\lambda$, $\forall n, m$ _ $M(\varepsilon,\lambda)$. A Menger space (X, F, t) is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 6: Self-maps A and S of a Menger space (X, F, t) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$.

Definition 7: [7] Self mappings A and S of a Menger space (X, F, t) are called compatible if $FASpn, SApn(x) \rightarrow 1$ for all $x > 0$, whenever $\{pn\}$ is a sequence in X such that $Apn=Spn \rightarrow u$, for some u in X, as $n \rightarrow \infty$.

Proposition 2: *Self mappings A and S of a Menger space (X, F, t) are compatible then* they are weakly compatible.

Example 1: Let (X, d) be a metric space where $X = [0, 2]$ and (X, F, t) be the induced Menger space with

$Fp, q(\varepsilon) = H(\varepsilon - d(p, q))$, $\forall p, q \in X$ and $\forall \varepsilon > 0$. Define self maps A and S as follows:

$Ax = 2 - x$, if $0 \leq x < 1$,

$= 2$, if $1 \leq x \leq 2$, and $Sx = x$, if $0 \leq x < 1$

$= 2$, if $1 \leq x \leq 2$.

Take $xn = 1 - 1/n$. Now,

$F_{AX_n}(\varepsilon) = \mathrm{H}(\varepsilon - (1/n))$ $\therefore \lim_{n\rightarrow\infty} F_{Ax_n}(\varepsilon)$ $\mathrm{H}(\varepsilon) = 1.$ hence $\mathrm{A}x_n \rightarrow 1$ *as* $n \rightarrow \infty$

Similarly, $\mathrm{S}x_n \rightarrow 1$ *as* $n \rightarrow \infty$.

Also, $\mathrm{F}ASx_n, SAx_n\ (\varepsilon) = \mathrm{H}(\varepsilon\ (1\text{-}1/\mathrm{n})$.

$$\lim_{n\rightarrow\infty} F_{ASx_n}\ ASx_n\ (\varepsilon) = H(\varepsilon - 1) \neq 1$$

Proposition 3: *In a Menger space (X, F, t), if $t(x, x) \geq x$ $\forall x \in [0, 1]$, then $t(a, b) = \min\{a, b\}$ $\forall a, b \in [0, 1]$.*

Lemma 1 [11]**:** *Let $\{p_n\}$ be a sequence in a Menger space (X, F, t) with continuous t -*Type equation here. *norm and $t(x, x)$ _ x. Suppose, for all $x \in [0, 1]$, $\exists k \in (0, 1)$ such that for all $x > 0$ and $n \in N$, $Fp_n, p_{n+1}(kx)$_$Fp_{n-1}, p_n(x)$. Then $\{p_n\}$ is a Cauchy sequence in X.*

Lemma 2: *Let (X, F, t) be a Menger space. If there exists k ∈ (0, 1) such that for p, q ∈ X, Fp, q(kx)_Fp, q(x). Then p = q.*

Theorem [7]: Let A, B, S and T be self maps of menger space (x, F, t) with continuous t-norm with t(x, x) ≥ **X** for all x∈[0,1] satisfying conditions
(a) A(x)⊑ T(X), B(x)⊑ S(X)
(b).∀p,q,∈ X,x>0 and for some k ∈(0,1)

$$F^3_{AX,ty}(kt) \geq \min\{F^3_{Lx,My}(t)F^3_{Ax,Lx}(t)F^3_{Ty,Lx}(T)F_{Ax,My}(2t)F_{Ty,Lx}(2t)F^2_{Ty,My}(t)$$

(c).the pair (A,S) and (B,T) are compatible.
(d) Sand T are continuous

Then A, B, S and T have a unique common fixed point.

3. MAIN RESULT

Theorem 3.1: Let A, B, S, T, T, L And M are self mapos on a complete Menger space (X, F, t) with t(a, a)≥a for all a∈ $[0,1], satisfying$

(a) L (X)⊑ ST(X) and M(X) ⊑ AB(X)
(b) AB =BA,ST=TS,LB=BL,MT=TM
(c) Either AB or L is continuous
(d) (L,AB) IS compatible and (M,ST) is weakly compatible..
(e) For all x, y∈ X ,k∈(0,1) , t>0

$$F^3_{ABX,Sty}(kt) \geq \min\{F^3_{Lx,My}(t)F^3_{ABx,Lx}(t)F^3_{STy,Lx}(T)F_{ABx,My}(2t)F_{STy,Lx}(2t)F^2_{STy,My}(t)$$

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof: Let $x_o \in X\ be\ any\ arbitrary\ point\ .Since\ L(X) \subseteq ST(X) and\ M(X) \subseteq AB(X)$.there exiost $x_1, x_2 \in X$ such that ABx_0 = Mx_1, and STx_1 = Lx_2, Mx_{2n-1} = ABx_{2n-2} and,y_{2n}= Lx_{2n}= STx_{2n-1} for n=1,2,3

Step (i): Now using (e) with x=x_{2n},y=x_{2n+1}, we have

$F^3_{y_{2n+1}y_{2n+2}}$(kt) = $F^3_{ABx_{2n,}}$stx_{2n+1}(kt)

$$\geq \min\{F^3_{Lx_{2n},Mx_{2n+1}}(t)\, F^3_{ABx_{2n},Lx_{2n}}(t), F^3_{STx_{2n},Mx_{2n+1}}(t),$$
$$F_{ABx_{2n},Mx_{2n+1}}(2t)\ F_{STx_{2n+1},Lx_{2n}}(2t)\ F^2_{STx_{2n+1},Mx_{2n+1}}(t)$$
$$\min\{F^3_{Lx_{2n},Mx_{2n+1}}(t)\, F^3_{ABx_{2n},Lx_{2n}}(t), F^3_{STx_{2n},Mx_{2n+1}}(t), F_{y_{2n},y_{2n+1}}(t)\ F_{y_{2n},,y_{2n+1}}y(2t)\ F^2_{STx_{2n+1},Mx_{2n+1}}(t)\}$$
$$\min\{F^3_{y_{2n},y_{2n+1}}(t)\, F^3_{,y_{2n+2}y_{2n+1}}(t), F^3_{,}y_{2n+1}y_{2n+1}t),$$
$$F_{y_{2n+2},,y_{2n+1}}y(2t)\ \ F_{y_{2n+1},,y_{2n+1}}y(2t)\ F^2_{,y_{2n+2}y_{2n+1}}(t)\}$$

$F_{y_{2n+1}y_{2n+2}}$(kt)≥ $F_{y_{2n},y_{2n+1}}(t)$

Therefore for all n we have $F_{y_n,,y_{n+1}}y(kt)$)≥ $F_{y_{n-1},y_n}(t)$

Hence it is Cauchy sequence in X, which is complete. Therefore $\{y_n\}$ converges to z belongs to X. Thus itsa subsequence also converges to Z.

Case (1): Let AB be continuous the AB (AB$X_{2N} \rightarrow ABz\ AND\ (AB)LX_{2N}) \rightarrow ABz$ As (l, AB) is semi compatible, then $(AB)LX_{2N}) \rightarrow Lz$ By uniqunsess of limit in menger space, we obatain ABz=Lz.

Step (ii): F^3_{ABZ}Mx_{2n+1}(kt) min{$F^3_{Lz,Mx_{2n+1}}$(t) $F^3_{ABz,Lz}$(t), $F^3_{STx_{2n},Mx_{2n+1}}$(t), $F_{ABz,,Mx_{2n+1}}(2t)$ $F_{STx,,Lz}(2t)$ $F^2_{STx_{2n+1},Mx_{2n+1}}$(t)

Taking limit, we get

F^3_{ABZ}Mx_{2n+1}(kt) ≥min{$F^3_{Lz,Mx_{2n+1}}$(t) $F^3_{ABz,z}$(t), $F^3_{z.z,}$(t), $F_{ABz,,Mx_{2n+1}}(2t)$ $F_{STx_{2n+1},,Lz}(2t)$ $F^2_{z,z,,}$(t)
≥min{$F^3_{Lz,z}$(t) 1,1$F_{ABz,,z}(2t)F_{z,ABZ,,Lz}(2t)$, 1}
≥min{$F^3_{ABz,z}$(t)

Hence ABZ=z.

Step (iii): AS M(X) ⊂AB(X), there exists a point v∈X such that z=ABz=Mv

By taking x=z and y=v in (e), we have

$F^3_{ABz,STv}(kt) \geq \min\{F^3_{Lz,Mv}\,(t)F^3_{ABz,z}\,(t)F^3_{STv,Lv}(T)F_{ABz,Mv}(2t)F_{STv,Lz}(2t)F^2_{STv,Mv}(t)$

Taking limit, we get

$$F^3_{z,STv}(kt) \geq \min\{F^3_{ABz,ABz}\,(t), 1, F^3_{stv,Mv}\,(t)F_{Mv,LMv}(T)F_{STv,ABz}(2t)F^2_{STv,Mv}(t)\}$$
$$\geq \min\{1,, 1, F^3_{stv,z}\,(t), 1, F_{STv,z}(2t)F^2_{STv,z}(t)\}$$

$F^3_{z,STv}(kt) \geq, F^3_{2,}(t),$

Hence z=STV= Mv

Step (iv): as (ST, M) is weakly compatible, M(STv)=ST(Mv), hence Mz=STz

By taking x=z and y=z in (e), we get

$$F^3_{ABz,STz}(kt) \geq \min\{F^3_{Lz,Mz}\,(t)F^3_{ABz,,Lz}\,(t)F^3_{STz,Mz}(T)F_{ABz,Mz}(2t)F_{STz,Lz}(2t)F^2_{STz,Mz}(t)$$
$$\geq \min\{F^3_{ABz,STz}\,(t), F^3_{ABz,STz}\,(t), F^3_{Lz,Lz}\,(t)F_{STz,ABz}(T)F_{ABz,STz}(2t)F^2_{Mz,Mz}(t)$$
$$\geq \min\{F^3_{ABz,STz}\,(t)1,1,, F^3_{Lz,Lz}\,(t)F_{ABz,STz}(2t), 1,\}$$
$$\geq \min\{F^3_{ABz,STz}\,(t)$$

Hence ABz= STz

Now, combine all the results we get z=ABz=STz=LZ=Mz. Thus z is the common fixed point of A, B, S, T, L AND M,.

Uniqueness: Let W (W$\neq Z$) be another common fixed point of A, B, S, T, L and M

Then w=ABw =STw =Lw=Mw.

$F^3_{ABz,,STw}(kt) \geq \min\{F^3_{Lz,Mw}\,(t)F^3_{ABz,Lz}\,(t)F^3_{STw,,Mx}(T)F_{ABz,Mw}(2t)F_{STw,Lz}(2t)F^2_{STw,Mw}(t)$

$F^3_{z,,w}(kt) \geq \min\{F^3_{z,w}\,(t)F^3_{z,z}\,(t)F^3_{w,,w}(T)F_{z,w}(2t)F_{w,z}(2t)F^2_{w,w}(t) \geq \{F^3_{z,w}\,(t)\}$

Hence z=w for all x, y∈ $X\ and\ t > 0.\ there\ fore\ z\ is\ the\ unique\ common\ fixed\ point\ of\ A, B, S, T, L\ AND\ M$
On taking B=T=I (identity maps).

Corollary 3.2: let A, S, Land M are self maps on a complete menger space (X, F, t) with t(x, x,)≥ $x\ for\ all\ x$ ∈[0,1] satisfying
(1).L(X)⊑s(X),M(X) ⊑ A(X)
(2).Either A or L is continous.
(3). (L, A) is compatible and (M, S) is weakly compatible.
(4). There exists k∈ (0,1) such that $F^3_{AX,ty}(kt) \geq \min\{F^3_{Lx,My}\,(t)F^3_{Ax,Lx}\,(t)F^3_{Ty,Lx}(T)F_{Ax,My}(2t)F_{Ty,Lx}(2t)F^2_{Ty,My}(t)$

For all p, q∈ X, $\beta \in (0,2) and\ x > 0$

Then A, S, L and M have a unique common fixed point in X.

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