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m-SERIES OF THE GENERALIZED DIFFERENCE EQUATION TO CIRCULAR FUNCTIONS

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ABSTRACT

We investigate the numerical-complete solution to certain type of generalized higher order difference equation to find the value of m-series to circular functions in the field of finite difference methods. We also give an example to illustrate the m-series.

Key words: *Complete solution, Circular function, Generalized difference operator, Numerical solution.*

AMS Subject classification: *39A70, 47B39, 39A10.*

1. INTRODUCTION

The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermodynamics), chemistry, biology, economics and control theory ([11], [12], [13], [14], [15]). In 1989, K.S.Miller and Ross [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The main definition of fractional difference equation (as done in [8]) is the ν fractional sum of $f(t)$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma_{(\nu)}} \sum_{s=a}^{t-\nu} \frac{\Gamma_{(t-s)}}{\Gamma_{(t-s-(\nu-1))}} f(s), \tag{1}$$

where $\nu > 0$. On the other hand, when $\nu = m$ is a positive integer, if we replace $f(t)$ by $u(k)$ and Δ by Δ_ℓ, (as given in definition 2.8 of [8]) then (1) becomes

$$u_{m(\ell)}(k) \equiv \Delta_\ell^{-m} = \sum_{r=m}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(m-1)}}{(m-1)!} u(k-r\ell) \tag{2}$$

where $(r-1)^{(m-1)} = (r-1)(r-2)\cdots(r-m+1)$ and $\left[\frac{k}{\ell}\right]$ is the integer part of $\frac{k}{\ell}$. Now (2) is very useful to derive many interesting results in a different way, such as the sum of the m^{th} partial sums to the n^{th} powers and the products of n consecutive terms of arithmetic and geometric progressions [6]. During the last decades several fractional sums for various functions have been investigated by numerous mathematicians (c.f.e.g, [1], [3], [9], [10] and the bibliography quoted there).

Let $\ell > 0$, $u(k)$ be real valued function on $[0,\infty)$, $u(k) = 0$ for all $k \in (-\infty, 0)$. Then, for $m \in \mathbb{N}(1)$, the m-series to $u(k)$ with respect to ℓ is defined as below:

$1-series;$ $\quad u_{1(\ell)}(k) = u(k-\ell) + u(k-2\ell) + \cdots + u\left(k - \left[\frac{k}{\ell}\right]\ell\right),$

$2-series;$ $\quad u_{2(\ell)}(k) = u_{1(\ell)}(k-\ell) + u_{1(\ell)}(k-2\ell) + \cdots + u_{1(\ell)}\left(k - \left[\frac{k}{\ell}\right]\ell\right),$ and in general, m-series ;

$$u_{m(\ell)}(k) = u_{(m-1)(\ell)}(k-\ell) + u_{(m-1)(\ell)}(k-2\ell) + \cdots + u_{(m-1)(\ell)}\left(k - \left[\frac{k}{\ell}\right]\ell\right).$$

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There are direct formulas for finding the value of the m-series to the k_ℓ^n functions $k^n, k_\ell^n, a^k, k^n a^k etc$ ([2], [4], [5], [6], [7]). If $u(k), u(k-\ell), \cdots$, and $u\left(k-\left[\frac{k}{\ell}\right]\ell\right)$ denote the amounts of infections of a disease in a body at the times $k, k-\ell, \cdots$, and $k-\left[\frac{k}{\ell}\right]\ell$ respectively , then $u_{m(\ell)}(k+m\ell)$ gives the total amount of infection of the disease for m-generations. To give proper medical treatment, it is necessary to find the exact value of m-series to the function $u(k)$ in the field of Health Science [7].

We find that the m-series of $u(k)$ with respect to ℓ is the $u_{m(\ell)}$ given in (2) and it is a numerical solution of the difference equation given by

$$\Delta_\ell^m v(k) = u(k), k \in [0,\infty), \ell > 0. \tag{3}$$

The complete solution, say $c_{m(\ell)}(k)$ of equation (3) is also a solution which provides the values of the m-series. Hence in this paper, we obtain the value of m-series to circular functions with respect to ℓ, since amount of infection of the disease is a circular function (increase and decrease with respect to medical treatment).

2. PRELIMINARIES

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be useful for further subsequent discussions. Throughout this paper, let $\ell > 0$, $k \in [0,\infty)$ is a variable, $j = k-\left[\frac{k}{\ell}\right]\ell$, $\mathbb{N}_\ell(j) = \{j, \ell+j, 2\ell+j, \cdots\}$ and $\mathbb{N}_1(j) = \mathbb{N}(j)$. c_j is constant for all $k \in \mathbb{N}_\ell(j)$ and for any positive integer m, we denote

$$c_m(\ell)(k) \equiv \Delta_\ell^{-m} u(k) \|_{(m-1)\ell+j}^{k} = \Delta_\ell^{-1}\left(\cdots \Delta_\ell^{-1}\left(\Delta_\ell^{-1} u(k)\,|_j^k\right)|_{\ell+j}^k \cdots\right)|_{(m-1)\ell+j}^k .$$

Definition 2.1: [5] *Let $u(k)$, $k \in [0,\infty)$ be a real valued function. The generalized difference operator Δ_ℓ on $u(k)$ is defined as;*

$$\Delta_\ell u(k) = u(k+\ell) - u(k),\ k \in [0,\infty),\ \ell \in (0,\infty), \tag{4}$$

and the inverse of Δ_ℓ on $u(k)$ is defined as,

$$\text{if } \Delta_\ell v(k) = u(k), \text{ then } v(k) = \Delta_\ell^{-1} u(k) + c_j. \tag{5}$$

In general,

$$\Delta_\ell^{-\nu} = \Delta_\ell^{-1}\left(\Delta_\ell^{-(\nu-1)}\right). \tag{6}$$

Lemma 2.2: *Let p and q be any two real numbers such that $p\ell$ and $q\ell$ are not integer multiple of 2π. Then, when $m=1$, equation (3) has solutions*

$$\Delta_\ell^{-1} \sin pk = \frac{\sin p(k-\ell) - \sin pk}{2(1-\cos p\ell)} + c_j \tag{7}$$

and

$$\Delta_\ell^{-1} \cos pk = \frac{\cos p(k-\ell) - \cos pk}{2(1-\cos p\ell)} + c_j \tag{8}$$

for $u(k) = \sin pk$ and $u(k) = \cos qk$ respectively .

Proof: Replacing u(k) by $\sin pk$ and $\cos pk$ in (4), we find that

$$\Delta_\ell \sin pk = (\cos p\ell - 1)\sin pk + \sin p\ell \cos pk, \tag{9}$$

and

$$\Delta_\ell \cos pk = (\cos p\ell - 1)\cos pk + \sin p\ell \sin pk. \tag{10}$$

Since Δ_ℓ is linear, i.e., $c\Delta_\ell u(k) = \Delta_\ell c u(k)$ and $(\cos p\ell - 1)$ and $\sin p\ell$ are constants, multiplying (9) by $(\cos p\ell - 1)$, (10) by $\sin p\ell$ and then subtracting the second resultant from the first one, we find that

$$\Delta_\ell[(\cos p\ell - 1)\sin pk - \sin p\ell \cos pk] = (2 - 2\cos p\ell)\sin pk. \tag{11}$$

Now (7) follows from (5) and dividing (11) by $2(1-\cos p\ell)$.

Similarly multiplying (9) by $\sin p\ell$, (10) by $(\cos p\ell - 1)$ and then adding them, we find that

$$\Delta_\ell[\sin p\ell \sin pk - (\cos p\ell - 1)\cos pk] = (2 - 2\cos p\ell)\cos pk. \tag{12}$$

Now (8) follows from (5) and dividing (12) by $2(1-\cos p\ell)$.

Lemma 2.3: *If $p\ell$ and $q\ell$ are not multiple of 2π, then*

$$\Delta_\ell^{-m}\sin pk = \sum_{t=0}^{m}\frac{m^{(t)}}{t!}\frac{\sin p(k-(m-t)\ell)}{2^m(1-\cos p\ell)^m} + c_j, \tag{13}$$

$$\Delta_\ell^{-m}\cos qk = \sum_{t=0}^{m}\frac{m^{(t)}}{t!}\frac{\cos q(k-(m-t)\ell)}{2^m(1-\cos q\ell)^m} + c_j \tag{14}$$

are closed form solutions of equation (3) when $u(k) = sinpk$, $cosqk$ respectively.

Proof: When $m=1$, (13) is followed from (7) and by induction on m, $m \ge 2$, we assume that,

$$\Delta_\ell^{-(m-1)}\sin pk = \sum_{t=0}^{m-1}\frac{(m-1)^{(t)}}{t!}\frac{\sin p(k-(m-1-t)\ell)}{2^{(m-1)}(1-\cos p\ell)^{(m-1)}} + c_j. \tag{15}$$

Since Δ_ℓ^{-1} is linear and $\cos p\ell$ is constant, from (7), we have

$$\Delta_\ell^{-1}\sin p(k-(m-1-t)\ell) = \frac{\sin p(k-(m-t)\ell) - \sin p(k-(m-1-t)\ell)}{2(1-\cos p\ell)}. \tag{16}$$

Since $\dfrac{(m-1)^{(r-1)}}{(r-1)!} + \dfrac{(m-1)^{(r)}}{r!} = \dfrac{m^{(r)}}{r!}$, (13) follows by taking Δ_ℓ^{-1} on (15), applying (16) and equating coefficients of $\sin(k-(m-t)\ell)$ for $t = 0,1,....,m$.

Similar argument gives the proof of (14).

Lemma 2.4: *Let $n \in N(1)$, $k \in [0,\infty)$ and p,q are constants. Then*

$$\sin^n pk = \begin{cases} \dfrac{1}{2^{n-1}}\displaystyle\sum_{r=0}^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}+r}\frac{n^{(r)}}{r!}\sin p(n-2r)k & \text{if } n \text{ is odd} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{r=0}^{\frac{n-2}{2}}(-1)^{\frac{n}{2}+r}\frac{n^{(r)}}{r!}\cos p(n-2r)k + \frac{n^{\left(\frac{n}{2}\right)}}{2\left(\frac{n}{2}\right)!} & \text{if } n \text{ is even.} \end{cases} \tag{17}$$

and

$$\cos^n qk = \begin{cases} \dfrac{1}{2^{n-1}}\displaystyle\sum_{r=0}^{\frac{n-1}{2}}\frac{n^{(r)}}{r!}\cos q(n-2r)k & \text{if } n \text{ is odd} \\ \dfrac{1}{2^{n-1}}\displaystyle\sum_{r=0}^{\frac{n-2}{2}}\frac{n^{(r)}}{r!}\cos q(n-2r)k + \frac{n^{\left(\frac{n}{2}\right)}}{2\left(\frac{n}{2}\right)!} & \text{if } n \text{ is even.} \end{cases} \tag{18}$$

Remark 2.5: *Hereafter we take* $P = p(n_1-2r_1)+q(n_2-2r_2)$ *and* $\overline{P} = p(n_1-2r_1)-q(n_2-2r_2)$ *and hence P and* $\overline{P}$ *are varying with respect to* n_1, n_2, r_1, r_2, p *and* q.

Corollary 2.6: *(i) If* n_1 *and* n_2 *are odd positive integers, then*

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!} \left\{\sin Pk + \sin \overline{P}k\right\}. \tag{19}$$

(ii) If n_1 is an odd positive integer and n_2 is an even positive integer, then

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} (-1)^{\frac{n_1-1}{2}+r_1} \frac{n_1^{(r_1)}}{r_1!} \left\{ \sum_{r_2=0}^{\frac{n_2-1}{2}} \frac{n_2^{(r_2)}}{r_2!} \left(\sin Pk + \sin \overline{P}k\right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \sin\left(\frac{P+\overline{P}}{2}\right)k \right\}. \tag{20}$$

(iii) If n_1 is an even positive integer and n_2 is an odd positive integer, then

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \sum_{r_2=0}^{\frac{n_2-1}{2}} \frac{n_2^{(r_2)}}{r_2!} \left\{ \sum_{r_1=0}^{\frac{n_1-1}{2}} (-1)^{\frac{n_1}{2}+r_1} \frac{n_1^{(r_1)}}{r_1!} \left(\cos Pk + \cos \overline{P}k\right) + \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \cos\left(\frac{P-\overline{P}}{2}\right)k \right\}. \tag{21}$$

(iv) If n_1 and n_2 are even positive integers, then

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \left\{ \left(\sum_{r_1=0}^{\frac{n_1-2}{2}} (-1)^{\frac{n_1}{2}+r_1} \frac{n_1^{(r_1)}}{r_1!} \left(\sum_{r_2=0}^{\frac{n_2-2}{2}} \frac{n_2^{(r_2)}}{r_2!} \left(\cos Pk + \cos \overline{P}k\right) \right.\right.\right.$$
$$\left.\left.\left. + \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \cos\left(\frac{P-\overline{P}}{2}\right)k \right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \cos\left(\frac{P+\overline{P}}{2}\right)k \right) + \frac{1}{2} \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \right\}. \tag{22}$$

Proof: The proof of (19), (20), (21) and (22) are obtained by combining (17) and (18) and using the properties of trigonometric functions.

Corollary 2.7: *(i) If* n_1 *and* n_2 *are odd positive integers, then*

$$\sin^{n_1} pk \sin^{n_2} qk = \frac{(-1)^{\frac{n_1+n_2-2}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1+r_2} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!} \left\{\cos Pk - \cos \overline{P}k\right\} \tag{23}$$

(ii) If n_1 is an odd positive integer and n_2 is an even positive integer, then

$$\sin^{n_1} pk \sin^{n_2} qk = \frac{(-1)^{\frac{n_1+n_2-1}{2}}}{2^{n_1+n_2-1}} \left\{ \sum_{r_1=0}^{\frac{n_1-1}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \sum_{r_2=0}^{\frac{n_2-2}{2}} (-1)^{r_2} \frac{n_2^{(r_2)}}{r_2!} \left(\sin Pk + \sin \overline{P}k\right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \sin\left(\frac{P+\overline{P}}{2}\right)k \right\} \tag{24}$$

(iii) If n_1 and n_2 are even positive integers, then

$$\sin^{n_1} pk \sin^{n_2} qk = \frac{(-1)^{\frac{n_1+n_2}{2}}}{2^{n_1+n_2-1}} \left\{ \left(\sum_{r_1=0}^{\frac{n_1-2}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \left(\sum_{r_2=0}^{\frac{n_2-2}{2}} \frac{n_2^{(r_2)}}{r_2!} \frac{\left(\cos Pk + \cos \overline{P}k\right)}{(-1)^{r_2}} \right.\right.\right.$$
$$\left.\left.\left. + \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \cos\left(\frac{P-\overline{P}}{2}\right)k \right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \cos\left(\frac{P+\overline{P}}{2}\right)k \right) + \frac{1}{2} \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \right\}. \tag{25}$$

Proof: The proof of (23), (24) and (25) are obtained by using (17) and the properties of trigonometric functions.

Corollary 2.8: *(i) If* n_1 *and* n_2 *are odd positive integers, then*

$$\cos^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!} \left\{ \cos Pk + \cos \overline{P}k \right\}. \tag{26}$$

(ii) If n_1 is an odd positive integer and n_2 is an even positive integer, then

$$\cos^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \left\{ \sum_{r=0}^{\frac{n_2-1}{2}} \frac{n_1^{(r_1)}}{r_1!} \left(\sum_{r_2=0}^{\frac{n_2-2}{2}} \frac{n_2^{(r_2)}}{r_2!} \left(\cos Pk + \cos \overline{P}k \right) \right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \cos\left(\frac{P+\overline{P}}{2} \right) k \right\}. \tag{27}$$

(iii) If n_1 and n_2 are even positive integers, then

$$\cos^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}} \left\{ \left(\left(\sum_{r_1=0}^{\frac{n_1-2}{2}} \frac{n_1^{(r_1)}}{r_1!} \left(\sum_{r_2=0}^{\frac{n_2-2}{2}} \frac{n_2^{(r_2)}}{r_2!} \left(\cos Pk + \cos \overline{P}k \right) \right. \right. \right. \right.$$

$$\left. \left. \left. \left. + \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \cos\left(\frac{P-\overline{P}}{2} \right) k \right) + \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \cos\left(\frac{P+\overline{P}}{2} \right) k \right) + \frac{1}{2} \frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!} \frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!} \right\}. \tag{28}$$

Proof: The proof of (26), (27) and (28) are obtained by using (18) and the properties of trigonometric functions.

3. MAIN RESULTS

In this section, we use the following notations: $L_{m-1} = \{1, 2, \ldots, m-1\}$, $0(L_{m-1}) = \{\phi\}$, ϕ is empty set, $1(L_{m-1}) = \{\{1\},\{2\},\cdots,\{m-1\}\}$, $2(L_{m-1}) = \{\{1,2\},\{1,3\},\cdots,\{1,m-1\}, \{2,3\},\cdots,\{2,m-1\},\cdots,\{m-2,m-1\}\}$. In general, $t(L_{m-1})$ = set of all subsets of size t from the set L_{m-1} such that if $\{m_1, m_2, \cdots, m_t\} \in t(L_{m-1})$ then $m_1 < m_2 < \cdots < m_t$, $(m-1)(L_{m-1}) = \{\{1, 2, \cdots, m-1\}\}$, $\wp(L_{m-1}) = \bigcup_{t=0}^{m-1} t(L_{m-1})$, power set of L_{m-1}, $\sum_{t=1}^{m-1} f(t) = 0$ for $m \le 1$, and $\prod_{i=2}^{t} f(i) = 1$ for $t \le 1$, and $\{m_t\} \in t(L_{m-1})$ means that $\{m_1, m_2, \cdots, m_t\} \in t(L_{m-1})$.

Theorem 3.1: [2] *Let* $m \in \mathbb{N}(2)$, $0 < \ell < k$. *If* $\Delta_\ell^{-m} u(k)$ *is any closed form solutions of equation (3), then for* $k \in [m\ell, \infty)$,

$$C_{m(\ell)(k)} \Big|_{(m-1)\ell+j}^{k} = \Delta_\ell^{-m} u(k) \Big|_{(m-1)\ell+j}^{k} + \sum_{t=1}^{m-1} \sum_{\{m_t\} \in t(L_{m-1})} (-1)^t (\Delta_\ell^{-m_1} u((m_1-1)\ell + j))$$

$$\times \frac{k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}} \prod_{i=2}^{t} \frac{((m_i-1)\ell + j)_\ell^{(m_i-m_{i-1})}}{(m_i - m_{i-1})!\ell^{m_i-m_{i-1}}} \Big|_{(m-1)\ell+j}^{k} \tag{29}$$

is the complete solution of equation (3).

Theorem 3.2: [2] *(m-series formula) Let* $m \in \mathbb{N}(2)$. *Then, for* $k \in [m\ell, \infty)$,

$$\sum_{r=m}^{[\frac{k}{\ell}]} \frac{(r-1)^{(m-1)}}{(m-1)!} u(k - r\ell) = \Delta_\ell^{-m} u(k) \Big|_{(m-1)\ell+j}^{k} + \sum_{t=1}^{m-1} \sum_{\{m_t\} \in t(L_{m-1})} (-1)^t (\Delta_\ell^{-m_1} u((m_1-1)\ell + j))$$

$$\times \frac{k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}} \prod_{i=2}^{t} \frac{((m_i-1)\ell + j)_\ell^{(m_i-m_{i-1})}}{(m_i - m_{i-1})!\ell^{m_i-m_{i-1}}} \Big|_{(m-1)\ell+j}^{k} \tag{30}$$

In which LHS of (30) gives m-series and RHS provides the value of the m-series to $u(k)$.

Remark 3.3: *Hereafter we denote* $\Pi(t)=\prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}$ *and* $P\ell$, $\overline{P}\ell$, $\left(\frac{P+\overline{P}}{2}\right)\ell$, $\left(\frac{P-\overline{P}}{2}\right)\ell$ *are not integer multiple of* 2π.

Theorem 3.4: *If* n_1 *and* n_2 *are odd positive integers, then the m-series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=\frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+m-1}}\left\{\sum_{r_1=0}^{\frac{n_1-1}{2}}\sum_{r_2=0}^{\frac{n_2-1}{2}}\sum_{r_3=0}^{m}\frac{n_1^{(r_1)}n_2^{(r_2)}m^{(r_3)}}{(-1)^{r_1+r_3}r_1!r_2!r_3!}\right.$$

$$\left.\times\left(\frac{\sin P(k-(m-r_3)\ell)}{(1-\cos P\ell)^m}+\frac{\sin\overline{P}(k-(m-r_3)\ell)}{(1-\cos\overline{P}\ell)^m}\right)\right\}\Big|_{(m-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+m_1-1}}\left\{\sum_{r_1=0}^{\frac{n_1-1}{2}}\sum_{r_2=0}^{\frac{n_2-1}{2}}\sum_{r_4=0}^{m_1}(-1)^{r_1+r_4}\frac{n_1^{(r_1)}}{r_1!}\frac{n_2^{(r_2)}}{r_2!}\frac{m_1^{(r_4)}}{r_4!}\right.$$

$$\left.\times\left(\frac{\sin P((r_4-1)\ell+j)}{(1-\cos P\ell)^{m_1}}+\frac{\sin\overline{P}((r_4-1)\ell+j)}{(1-\cos\overline{P}\ell)^{m_1}}\right)\right\}\frac{\Pi(t)k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}}\Big|_{(m-1)\ell+j}^{k}\qquad(31)$$

Proof: The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\cos^{n_2}qk$ in theorem (3.2) and applying equation (19) on lemma (2.3).

Remark 3.5: *When* $n_2=0$ *in (31) we will get* $\Delta_\ell^{-m}\sin^{n_1}pk$ *and* when $n_1=0$ in (31) we will get $\Delta_\ell^{-m}\cos^{n_2}pk$.

The following example illustrates a 4-series to $\sin^3 6k\cos^3 5k$,

Example 3.6: *Consider the case* $m=4$, $p=6$, $q=5$, $n_1=3$, $n_2=3$, $P=\left(6(3-2r_1)+5(3-2r_2)\right)$ *and* $\overline{P}=\left(6(3-2r_1)-5(3-2r_2)\right)$. *In this case,* $L_3=\{1,2,3\}, 1(L_3)=\{\{1\},\{2\},\{3\}\}, 2(L_3)=\{\{1,2\},\{1,3\},\{2,3\}\}, 3(L_3)=\{\{1,2,3\}\}$ *and (31) becomes*

$$\sum_{r=4}^{[\frac{k}{\ell}]}\frac{(r-1)^{(3)}}{(3)!}\sin^3 6(k-r\ell)\cos^3 5(k-r\ell)=\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+4-1}}\frac{4^{(r_3)}}{r_3!}\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{4}(-1)^{r_1+r_3}\right.$$

$$\left.\times\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\left(\frac{\sin P(k-(4-r_3)\ell)}{(1-\cos P\ell)^4}\frac{\sin\overline{P}(k-(m-r_3)\ell)}{(1-\cos\overline{P}\ell)^4}\right)\right\}\Big|_{(4-1)\ell+j}^{k}$$

$$+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}\frac{(-1)^{\frac{3-1+t}{2}}}{2^{3+3+m_1-1}}\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_4=0}^{m_1}(-1)^{r_1+r_4}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{m_1^{(r_4)}}{r_4!}\right.$$

$$\left.\times\left(\frac{\sin P((r_4-1)\ell+j)}{(1-\cos P\ell)^{m_1}}+\frac{\sin\overline{P}((r_4-1)\ell+j)}{(1-\cos\overline{P}\ell)^{m_1}}\right)\right\}\frac{\Pi(t)k_\ell^{(4-m_t)}}{(4-m_t)!\ell^{m-m_t}}\Big|_{(4-1)\ell+j}^{k}.\qquad(32)$$

The five summation expression of (32) can be obtained by adding the sums corresponds to;

$$\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{1}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+1-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{1^{(r_3)}}{r_3!}\left(\frac{\sin P(j-(1-r_3)\ell)}{(1-\cos P\ell)^1}+\frac{\sin \overline{P}(j-(1-r_3)\ell)}{(1-\cos \overline{P}\ell)^1}\right)\frac{k_\ell^{(3)}}{3!\ell^3}\right.$$

$$+\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{2}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+2-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{2^{(r_3)}}{r_3!}\left(\frac{\sin P((\ell+j)-(2-r_3)\ell)}{(1-\cos P\ell)^2}+\frac{\sin \overline{P}((\ell+j)-(2-r_3)\ell)}{(1-\cos \overline{P}\ell)^2}\right)\right\}\frac{k_\ell^{(2)}}{2!\ell^2}$$

$$+\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{3}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+3-1}}\left(\frac{\sin P((2\ell+j)-(3-r_3)\ell)}{(1-\cos P\ell)^3}+\frac{\sin \overline{P}((2\ell+j)-(2-r_3)\ell)}{(1-\cos \overline{P}\ell)^3}\right)\right\}\frac{k_\ell^{(1)}}{1!\ell^1}$$

corresponds to $2(L_3)$;

$$\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{1}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+1-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{1^{(r_3)}}{r_3!}\left(\frac{\sin P(j-(1-r_3)\ell)}{(1-\cos P\ell)^1}+\frac{\sin \overline{P}(j-(1-r_3)\ell)}{(1-\cos \overline{P}\ell)^1}\right)\right\}\frac{k_\ell^{(2)}}{2!\ell^2}\frac{(\ell+j)_\ell^{(1)}}{\ell}$$

$$+\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{1}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+1-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{1^{(r_3)}}{r_3!}\left(\frac{\sin P(j-(1-r_3)\ell)}{(1-\cos P\ell)^1}+\frac{\sin \overline{P}(j-(1-r_3)\ell)}{(1-\cos \overline{P}\ell)^1}\right)\right\}\frac{k_\ell^{(1)}}{1!\ell^1}\frac{(2\ell+j)_\ell^{(2)}}{2!\ell^2}$$

$$+\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{2}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+2-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{2^{(r_3)}}{r_3!}\left(\frac{\sin P((\ell+j)-(2-r_3)\ell)}{(1-\cos P\ell)^2}\right.\right.$$

$$\left.\left.+\frac{\sin \overline{P}((\ell+j)-(2-r_3)\ell)}{(1-\cos \overline{P}\ell)^2}\right)\right\}\frac{k_\ell^{(1)}}{1!\ell^1}\frac{(2\ell+j)_\ell^{(1)}}{1!\ell^1}$$

and to $3(L_3)$:

$$\left\{\sum_{r_1=0}^{\frac{3-1}{2}}\sum_{r_2=0}^{\frac{3-1}{2}}\sum_{r_3=0}^{2}\frac{(-1)^{\frac{3-1}{2}+r_1+r_3}}{2^{3+3+1-1}}\frac{3^{(r_1)}}{r_1!}\frac{3^{(r_2)}}{r_2!}\frac{1^{(r_3)}}{r_3!}\left(\frac{\sin P(j-(1-r_3)\ell)}{(1-\cos P\ell)^1}+\right.\right.$$

$$\left.\left.\frac{\sin \overline{P}(j-(1-r_3)\ell)}{(1-\cos \overline{P}\ell)^1}\right)\right\}\frac{k_\ell^{(1)}}{\ell}\frac{(\ell+j)_\ell^{(1)}}{\ell}\frac{(2\ell+j)_\ell^{(1)}}{\ell}.$$

Theorem 3.7: *If* n_1 *is an odd positive integer and* n_2 *is an even positive integer, then the m-series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=\frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+m-1}}\left\{\sum_{r_1=0}^{\frac{n_1-1}{2}}\sum_{r_3=0}^{m}(-1)^{r_1+r_3}\frac{n_1^{(r_1)}}{r_1!}\frac{m^{(r_3)}}{r_3!}\right.$$

$$\times\left(\sum_{r_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(r_2)}}{r_2!}\left(\frac{\sin P(k-(m-r_2)\ell)}{(1-\cos P\ell)^m}+\frac{\sin \overline{P}(k-(m-r_3)\ell)}{(1-\cos \overline{P}\ell)^m}\right)\right)$$

$$+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\left(\frac{\sin\left(\frac{P+\overline{P}}{2}\right)(k-(m-r_3)\ell)}{(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell)^m}\right)\Bigg\}\Big|_{(m-1)\ell+j}^{k}+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+m-1}}\left\{\sum_{r_1=0}^{\frac{n_1-1}{2}}\sum_{r_4=0}^{m}(-1)^{r_1+r_4}\right.$$

$$\frac{n_1^{(r_1)}}{r_1!}\frac{m_1^{(r_4)}}{r_4!}\left(\sum_{r_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(r_2)}}{r_2!}\left(\frac{\sin P((r_4-1)\ell+j)}{(1-\cos P\ell)^{m_1}}+\frac{\sin\overline{P}((r_4-1)\ell+j)}{(1-\cos\overline{P}\ell)^{m_1}}\right)\right)$$

$$\left.+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\left(\frac{\sin\left(\frac{P+\overline{P}}{2}\right)((r_4-1)\ell+j)}{(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell)^{m_1}}\right)\right\}\frac{\Pi(t)\times k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}}\Big|_{(m-1)\ell+j}^{k} \tag{33}$$

Proof: The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\cos^{n_2}qk$ in theorem (3.2) and applying equation (20) on Lemma (2.3).

Theorem 3.8: *If* n_1 *is an even positive integer and* n_2 *is an odd positive integer then the m-series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\sin^{n_1}p(k-r\ell)\cos^{n_2}q(k-r\ell)=\frac{1}{2^{n_1+n_2+m-1}}\left\{\sum_{r_2=0}^{\frac{n_2-1}{2}}\sum_{r_3=0}^{m}(-1)^{r_3}\frac{n_2^{(r_2)}}{r_2!}\frac{m^{(r_3)}}{r_3!}\right.$$

$$\left(\sum_{r_1=0}^{\frac{n_1-2}{2}}(-1)^{\frac{n_1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\frac{\cos P(k-(m-r_3)\ell)}{(1-\cos P\ell)^m}+\frac{\cos\overline{P}(k-(m-r_3)\ell)}{(1-\cos\overline{P}\ell)^m}\right)\right)$$

$$\left.+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\left(\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(k-(m-r_3)\ell)}{(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell)^m}\right)\right\}\Big|_{(m-1)\ell+j}^{k}+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}\frac{(-1)^t}{2^{n_1+n_2+m_1-1}}$$

$$\times\left\{\sum_{r_2=0}^{\frac{n_2-1}{2}}\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{n_2^{(r_2)}}{r_2!}\frac{m_1^{(r_4)}}{r_4!}\left(\sum_{r_1=0}^{\frac{n_1-2}{2}}(-1)^{\frac{n_1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\frac{\cos P((r_4-1)\ell+j)}{(1-\cos P\ell)^{m_1}}+\frac{\cos\overline{P}((r_4-1)\ell+j)}{(1-\cos\overline{P}\ell)^{m_1}}\right)\right)\right.$$

$$\left.+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\left(\frac{\cos\left(\frac{P-\overline{P}}{2}\right)((r_4-1)\ell+j)}{(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell)^{m_1}}\right)\right\}\frac{\Pi(t)k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}}\Big|_{(m-1)\ell+j}^{k}. \tag{34}$$

Proof: The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\cos^{n_2}qk$ in theorem (3.2) and applying equation (21) on Lemma (2.3).

Theorem 3.9: *If* n_1 *and* n_2 *are even positive integers then the m-series to* $\sin^{n_1}p(k)\cos^{n_2}q(k)$ *is given by*

$$\sum_{r=m}^{[\frac{k}{\ell}]}\frac{(r-1)^{(m-1)}}{(m-1)!}\sin^{n_1}p(k-r\ell)\sin^{n_2}q(k-r\ell)=\frac{(-1)^{\frac{n_1+n_2}{2}}}{2^{n_1+n_2+m-1}}\left[\sum_{r_3=0}^{m}(-1)^{r_3}\frac{m^{(r_3)}}{r_3!}\left(\sum_{r_1=0}^{\frac{n_1-2}{2}}\right.\right.$$

$$(-1)^{r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\sum_{r_2=0}^{\frac{n_2-2}{2}}(-1)^{r_2}\frac{n_2^{(r_2)}}{r_2!}\left(\frac{\cos P(k-(m-r_3)\ell)}{(1-\cos P\ell)^m}+\frac{\cos\overline{P}(k-(m-r_3)\ell)}{(1-\cos\overline{P}\ell)^m}\right)\right.$$

$$\left.\left.+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(k-(m-r_3)\ell)}{(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell)^m}\right)+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(k-(m-r_3)\ell)}{(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell)^m}\right)$$

$$\left.+2^{m-1}\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{k_\ell^{(m)}}{m!\ell^m}\right]\Big|_{(m-1)\ell+j}^{k}+\sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}\frac{(-1)^{\frac{n_1+n_2}{2}+t}}{2^{n_1+n_2+m_1-1}}\left[\sum_{r_4=0}^{m_1}(-1)^{r_4}\frac{m_1^{(r_4)}}{r_4!}\right.$$

$$+\left(\sum_{r_1=0}^{\frac{n_1-2}{2}}(-1)^{r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\sum_{r_2=0}^{\frac{n_2-2}{2}}(-1)^{r_2}\frac{n_2^{(r_2)}}{r_2!}\left(\frac{\cos P((r_4-1)\ell)}{(1-\cos P\ell)^{m_1}}+\frac{\cos\overline{P}((r_4-1)\ell+j)}{(1-\cos\overline{P}\ell)^{m_1}}\right)\right.\right.$$

$$\left.\left.+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)((r_4-1)\ell+j)}{(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell)^{m_1}}\right)+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)((r_4-1)\ell+j)}{(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell)^{m_1}}\right)$$

$$\left.+2^{m_1-1}\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{k_\ell^{(m_1)}}{m_1!\ell^{m_1}}\right]\frac{\Pi(t)k_\ell^{(m-m_t)}}{(m-m_t)!\ell^{m-m_t}}\Big|_{(m-1)\ell+j}^{k}. \tag{35}$$

Proof: The proof is obtained by replacing $u(k)$ by $\sin^{n_1}pk\cos^{n_2}qk$ in theorem (3.2) and applying equation (22) on lemma (2.3).

Remark 3.10: $n_2=0$ *in (35) gives* $\Delta_\ell^{-m}\sin^{n_1}pk$ *and* $n_1=0$ yields $\Delta_\ell^{-m}\cos^{n_2}pk$.

Remark 3.11: *Similarly, using Corollaries (2.7) and (2.8), Lemma (2.3) and Theorem(3.2), one can obtain m-series with respect to* ℓ *to the functions* $\cos^{n_1}pk\cos^{n_2}qk$ *and* $\sin^{n_1}pk\sin^{n_2}qk$.

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